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# *THE PARAMETRIC REPRESENTATION OF THE NEIGHBORHOOD OF A SINGULAR POINT OF AN ANALYTIC SURFACE.*

By C. W. M. Black.

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## INTRODUCTION.

### A. — Outline of Kobb's Treatment of the Problem.

The problem of the representation, by a finite number of parametric formulae in two variables, of the neighborhood of a singular point of an algebraic surface is considered and alleged to be solved in an article "Sur la théorie des fonctions algébriques de deux variables," * by Gustav Kobb. A brief outline of Kobb's method follows: —

1. *Treatment of the Original Singular Point.* 1) Let the equation of the surface be written in the form

$$F(x, y, z) = 0,$$

where $F$ is a function of the three independent variables $x, y, z$ analytic in the point $x = a, y = b, x = c$. The function $F$ is transformed by means of a change of axes to the form

$$\Phi(\xi, \eta, \zeta) \equiv (\xi, \eta, \zeta)_m + (\xi, \eta, \zeta)_{m+1} + \ldots\ldots = 0 \qquad \text{(a)}$$

where the expression $(\xi, \eta, \zeta)_n$ is a homogeneous polynomial of degree $n$, the resulting surface (a) having the singular point considered at the origin, while the function $(\xi, \eta, \zeta)_m$ is of a form convenient for later treatment.

2) By the quadratic transformation

$$\xi = \tau\zeta, \qquad \eta = \sigma\zeta,$$

$$\left.\begin{aligned} \Phi(\xi, \eta, \zeta) &= \zeta^m[(\tau, \sigma, 1)_m + \zeta(\tau, \sigma, 1)_{m+1} + \ldots\ldots] \\ &= \zeta^m[\phi(\tau, \sigma) + \zeta\chi(\tau, \sigma) + \ldots\ldots\ldots] \\ &= \zeta^m\Phi(\tau, \sigma, \zeta) \end{aligned}\right\} \qquad \text{(b)}$$

* Journal de mathématiques pures et appliquées, 4th Series, Vol. VIII. (1892), p. 385.

and the neighborhood of the original point is represented by the neighborhood of the curve

$$\phi(\tau, \sigma) = 0, \qquad \zeta = 0, \tag{c}$$

on the surface

$$\bar{\Phi}(\tau, \sigma, \zeta) = 0.$$

3) The neighborhood of the curve (c) is included in the domains of a finite number of points which are

*a.* regular points of the curve (c), the domain of each being represented by a single power series

$$\tau = p(\sigma, \zeta); \tag{d}$$

*b.* critical points of the curve (c), the domain of each being represented by an equation of the form

$$\tau^{\mu} + \bar{p}_1(\sigma, \zeta)\tau^{\mu-1} + \ldots\ldots + \bar{p}_{\mu-1}(\sigma, \zeta)\tau + \bar{p}_{\mu}(\sigma, \zeta) = 0; \tag{e}$$

*c.* points at an infinite distance on the curve (c), the domain of each being represented by an equation of form (d) or (e) in the variables $\tau_1$, $\sigma_1$, $\eta$, where

$$\frac{\tau}{\sigma} = \tau_1, \qquad \frac{1}{\sigma} = \sigma_1, \qquad \zeta\sigma = \eta.$$

4) The selection of the points in 3) depends upon the character of the curve

$$\phi(\tau, \sigma) = 0.$$

*a.* If $\phi$ is irreducible, all points of class 3) *b* are first taken, then all points of class 3) *c*, these being regular; finally a finite number of points of class 3) *a*. Here, all the points selected, if singular, are of order less than $m$.

*b.* If $\phi$ is reducible, but contains no multiple factors, the same selection of points holds as in *a*, but there may occur a singular point of order $m$.

*c.* If $\phi$ contains multiple factors, all critical points of the curves corresponding to any factor, together with all points of intersection of two different factors, are first taken, then all points of class 3) *c*, these being possibly singular; finally, a finite number of regular points of the several curves corresponding to the different factors of $\phi$, these last points being possibly singular points of the surface. In this case, there may occur a number of singular points of order $m$.

2. *Treatment of Points Determined in* 1. The same treatment as in 1 is applied to each of these points and to each of the corresponding

resulting points in turn, so long as they are singular. If after a finite number of such processes, all the resulting points are regular, then by combining the results it is assumed that the neighborhood of the original point is represented by the domains of a finite number of regular points, and so by a finite number of parametric formulae as desired.

3. *Proof that a Finite Number of the Processes of* 1 *will be Sufficient to make all Points in* 2 *Regular.* Starting with the surface

$$f(u, v, w) = 0, \tag{f}$$

in which the singular point considered is at the origin, the transformations in 1, 1) and 2) are combined in the form

$$\left.\begin{aligned} u &= (\alpha_1\tau + \beta_1\sigma + \gamma_1)\zeta \\ v &= (\alpha_2\tau + \beta_2\sigma + \gamma_2)\zeta \\ w &= (\alpha_3\tau + \beta_3\sigma + \gamma_3)\zeta \end{aligned}\right\} \tag{g}$$

We can assume that

$$\gamma_2 \neq 0, \qquad \gamma_3 \neq 0$$

by making, if necessary, upon $f(u, v, w)$ a suitable homogeneous linear transformation. Then the next set of transformations, in 2, can be expressed in the form

$$\left.\begin{aligned} \tau &= (\alpha_1'\tau_1 + \beta_1'\sigma_1 + \gamma_1')\zeta_1 \\ \sigma &= (\alpha_2'\tau_1 + \beta_2'\sigma_1 + \gamma_2')\zeta_1 \\ \zeta &= (\alpha_3'\tau_1 + \beta_3'\sigma_1 + \gamma_3')\zeta_1 \end{aligned}\right\} \tag{h}$$

in which $\gamma_3' \neq 0$,* and the later sets of transformations are of the same type with the corresponding $\gamma_3$'s:

$$\gamma_3'' \neq 0, \qquad \gamma_3''' \neq 0, \ldots\ldots \text{etc.}$$

So we consider a succession of transformations of type (g), which give a succession of surfaces with multiple points each of order $m$. These transformations will combine in the form

$$\left.\begin{aligned} u &= [\gamma_1\gamma_3'\gamma_3'' \ldots\ldots \gamma_3^{(r)} + (\tau_r, \sigma_r, \zeta_r)]\zeta_r = [\Gamma_1 + (\tau_r, \sigma_r, \zeta_r)]\zeta_r \\ v &= [\gamma_2\gamma_3'\gamma_3'' \ldots\ldots \gamma_3^{(r)} + (\tau_r, \sigma_r, \zeta_r)]\zeta_r = [\Gamma_2 + (\tau_r, \sigma_r, \zeta_r)]\zeta_r \\ w &= [\gamma_3\gamma_3'\gamma_3'' \ldots\ldots \gamma_3^{(r)} + (\tau_r, \sigma_r, \zeta_r)]\zeta_r = [\Gamma_3 + (\tau_r, \sigma_r, \zeta_r)]\zeta_r \end{aligned}\right\} \text{(i)}$$

where the symbol $(\tau_r, \sigma_r, \zeta_r)$ represents in the expression in which it occurs all of the variable terms, and $\Gamma_2 \neq 0$, $\Gamma_3 \neq 0$.

* To secure this, Kobb makes unwarranted use of a quadratic transformation, which, however, might be replaced by a homogeneous linear transformation. He also overlooks one class of transformations which will arise (see 4).

Next, as $f(u, v, w)$ can be supposed to be irreducible, we have a relation of the form

$$L(u, v, w) f(u, v, w) + M(u, v, w) \frac{\partial}{\partial u}[f(u, v, w)] = \chi(v, w)$$
$$= (v, w)_\lambda + (v, w)_{\lambda+1} + \ldots\ldots + (v, w)_n \not\equiv 0. \qquad \text{(j)}$$

Now it is shown that the first member of equation (j) becomes divisible by $\zeta_r^{(m-1)(r+1)}$ after the substitutions (i), and the establishment of an upper limit for the power of $\zeta_r$ which can then be taken out as a factor of the function resulting from $\chi(v, w)$, will secure a corresponding limit for $r$, as is needed to finish the proof.

## B. — Critique of Kobb's Analysis.

We now show in what respects Kobb's method and proof are at fault. Some of these errors are noted in a memoir "Sulla riduzione delle singolarità puntuali delle superficie algebriche dello spazio ordinario per trasformazioni quadratiche," by Beppo Levi.*

4. Kobb overlooks in his succession of transformations of type (g) the occurrence of transformations which arise from 1, 3), *c*. These are equivalent to

$$\left.\begin{aligned} \xi &= \tau_1 \eta \\ \eta &= \eta \\ \zeta &= \sigma_1 \eta \end{aligned}\right\}$$

and here the number corresponding to $\gamma_0'$ of (h) is zero; so that the proof, even if correct in other respects, would fail to cover all the cases involved.†

5. Without specific discussion of several unwarranted assumptions of Kobb,‡ we show by an example the failure of his proof for the upper limit of the exponent of the power of $\zeta_r$ to be taken out as a factor of $\chi(v, w)$ in (j) under the substitution (i). Let the given surface be

$$f = u^2 - 2uw - v^2 + 2vw + uvw - vw^2 - uw^2 + w^3 = 0. \qquad \text{(k)}$$

Here,

$$\chi(v, w) = (4 + w^2)(w - v)^2.$$

The curve

$$\phi(u, v) = u^2 - 2u - v^2 + 2v = 0$$

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* Annali di matematica, Series 2, Vol. XXVI. (1897), p. 219.

† Cf. Levi, l. c., p. 224. ‡ Cf. Levi, l. c., pp. 225–6.

has a singular point at

$$u = 1, \qquad v = 1.$$

So the first transformation is

$$\left.\begin{aligned} u &= (\tau + 1)\,\zeta \\ v &= (\sigma + 1)\,\zeta \\ w &= \qquad \zeta \end{aligned}\right\} \qquad \text{(l)}$$

which, applied to (k), gives

$$\zeta^2(\tau^2 - \sigma^2 + \tau\sigma\zeta) = 0 \qquad \text{(m)}$$

and

$$\chi(v, w) = \zeta^2\sigma^2(\zeta^2 + 4). \qquad \text{(n)}$$

Now the set of transformations to which Kobb is naturally led in this case is the following: — *

$$\left.\begin{array}{lll} \tau = \tau_1\,\zeta_1 & \sigma = \sigma_1\,\zeta_1 & \zeta = \zeta_1 \\ \tau_1 = \tau_2\,\zeta_2 & \sigma_1 = \sigma_2\,\zeta_2 & \zeta_1 = \zeta_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}\right\}$$

whence

$$\tau = \tau_r\,\zeta_r^r, \qquad \sigma = \sigma_r\,\zeta_r^r, \qquad \zeta = \zeta_r.$$

But this substitution in (n) gives

$$\chi(v, w) = \zeta_r^{2r+2}\,\sigma_r^2\,(\zeta_r^2 + 4)$$

in which the exponent of $\zeta_r$ increases indefinitely with $r$.

6. In the case in which the curve

$$\phi(\tau, \sigma) = 0$$

has multiple factors, the regular points of such factors taken in 1, 4) *c* are possibly singular points of the surface, whose domains are represented by equations of form (e). When a further quadratic transformation is applied to such a point, we are not warranted in assuming that the resulting developments will represent the whole of the domain of the point considered.† Kobb makes this assumption in proposing to

* This set, combined with the transformation (l), possesses all the properties required by Kobb in (g), (h), and (i); its appearance here invalidates his proof. It can easily be shown, moreover, that the most general set of transformations which he could use in this case would produce the same condition as shown here.

† The development about the point first considered, to begin with, is a relation *im Kleinen;* it becomes, however, on passing to the later transformations, a relation

use in 2 only the set of points determined in 1. We are not warranted, either, in assuming that, when a reduction of singularity arises from the appearance of a term of lower degree in a different variable from that with reference to which the first development is derived, the resulting development will hold throughout the same region as the first development. As an example consider the surface

$$\tau^2 + \sigma\zeta - \zeta = 0.$$

Regarded as a development for $\tau$, its coefficients converge for all finite values of $\sigma$ and $\zeta$; but when we develop for $\zeta$,

$$\zeta = \frac{\tau^2}{1-\sigma},$$

and the resulting series converges only when

$$|\sigma| < 1.$$

7. From geometrical considerations we should not expect the quadratic transformation used to resolve the singularity in all cases. In ordinary space the transformation

$$\xi = \tau\zeta, \qquad \eta = \sigma\zeta,$$

will transform in a one-to-one manner, without change of the $\zeta$ coördinate, all points except those in the $\zeta = 0$ plane. Now in the surface from (m),

$$\tau^2 - \sigma^2 + \tau\sigma\zeta = 0,$$

all points in the $\zeta$-axis are singular, and whatever the reduction that may be secured for the origin, there will be within the neighborhood of the origin points whose singularity is not reduced. The same considerations would be seen to apply if we had any space curve as a singular line.

Levi, in the article previously mentioned, does not attempt a proof of the entire proposition, but directs his work toward establishing by geometrical considerations the reduction of the singularity, making exception, however, of certain cases,* which are closely related to the one considered in 7.

Having thus considered the failure of Kobb to establish the proposition even for the general case of an algebraic surface, we shall, in the

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*im Grossen*, the limit to the number of points taken being determined by finding the extent of the domain of each; while the developments about the later points give relations *im Kleinen*, as far as the first point is concerned.

* Cf. Levi, l. c. p. 227. Cf. also a second paper by Levi, Atti R. Acc. Sci. Torino, Vol. XXXIII., 5 Dec, 1897.

present article, supply the deficiency, and treat at once the more general case of an analytic surface, *i. e.*, the case that the function $F(x, y, z)$ is not merely a polynomial, but is any analytic function which vanishes at the point $(a, b, c.)$

## § 1

### A. — The Fundamental Theorem.

1. The theorem, the proof of which forms the subject of this article, is the following.

Theorem: *Let $F(x, y, z)$ be a function such that*

1) *$F(x, y, z)$ is analytic in the three independent variables in the neighborhood of the point $x = a$, $y = b$, $z = c$;*

2) $F(a, b, c) = 0$;

3) $\left(\frac{\partial F}{\partial x}\right)_{(a, b, c)} = \left(\frac{\partial F}{\partial y}\right)_{(a, b, c)} = \left(\frac{\partial F}{\partial z}\right)_{(a, b, c)} = 0$;

*then we can represent all values of $(x, y, z)$ satisfying the equation*

$$F(x, y, z) = 0$$

*and lying in the neighborhood of the point $(a, b, c)$:*

$$|x - a| < \delta, \qquad |y - b| < \delta, \qquad |z - c| < \delta,$$

*by a finite number of parametric formulae of the following type:*

$$\left.\begin{aligned} x &= \phi_\rho(u, v) \\ y &= \psi_\rho(u, v) \\ z &= \chi_\rho(u, v) \end{aligned}\right\} \qquad \rho = 1, 2, \ldots\ldots \bar{\rho}, \qquad (A)$$

*where $\phi_\rho$, $\psi_\rho$, $\chi_\rho$ are analytic in the arguments $(u, v)$ throughout a certain region; further for each set of values of $(x, y, z)$, the values $(0, 0, 0)$ excepted, there corresponds for at least one value of $\rho$ a pair of values $(u, v)$ lying within the region in which the functions $\phi_\rho$, $\psi_\rho$, $\chi_\rho$ are considered, and for any value of $\rho$ for which this is the case, there corresponds no second pair of values. To the set of values $(0, 0, 0)$ corresponds at least one, and in general an infinite number of pairs of values $(u, v)$ for every value of $\rho$.*

2. *Explanation of Symbols.* The symbol $(x, y, z, \ldots\ldots)_n$ indicates, in the expression in which it appears, the total collection of terms of degree $n$ in the arguments taken together, which belong to that expression.

A functional sign expressed by means of a letter will always represent an analytic function.

The symbol $E(x, y, z, \ldots)$ will always represent a function which is analytic at the point $(0, 0, 0 \ldots)$ and for which $E(0, 0, 0 \ldots) \neq 0$. If written with a subscript, as $E_r(x, y, z, \ldots)$ it represents a particular function of the class; if without a subscript, it represents a general function of the class; so that two functions $E(x, y, z, \ldots)$ both expressed by the same symbol, need not be equal to each other.

## B. — The Transformations.

3. The equation

$$F(x, y, z) = 0$$

can be transformed to the form

$$\Phi(\xi, \eta, \zeta) = (\xi, \eta, \zeta)_m + (\xi, \eta, \zeta)_{m+1} + \ldots = 0$$

where

1) $m \geqq 2$,

2) the polynomial

$$(\xi, \eta, 1)_m = \phi(\xi, \eta)$$

contains the term $\xi^m$,

3) the points in which the curves corresponding to the irreducible factors of $\phi(\xi, \eta)$ cut the line at infinity shall be distinct from each other and from the point in which the line $\xi = 0$ cuts that line.

To do this, we first make the transformation

$$x = u + a, \qquad = v + b, \qquad z = w + c,$$

thus obtaining

$$F(x, y, z) = f(u, v, w) = (u, v, w)_m + (u, v, w)_{m+1} + \ldots$$

Here, $m \geqq 2$, the singularity now being at the origin. Next we make a linear homogeneous transformation with non-vanishing determinant,

$$\left.\begin{aligned} u &= \alpha_1 \xi + \beta_1 \eta + \gamma_1 \zeta \\ v &= \alpha_2 \xi + \beta_2 \eta + \gamma_2 \zeta \\ w &= \alpha_3 \xi + \beta_3 \eta + \gamma_3 \zeta \end{aligned}\right\} \qquad (1)$$

with the result:

$$f(u, v, w) = \Phi(\xi, \eta, \zeta) = (\xi, \eta, \zeta)_m + (\xi, \eta, \zeta)_{m+1} + \ldots = 0.$$

For this equation, conditions 2) and 3) can be secured, as is readily seen by a proper choice of the coefficients in transformation (1).

The surface $\Phi = 0$ corresponds in the neighborhood considered, point for point, to the surface $F = 0$, and thus it is only necessary to prove the theorem for $\Phi = 0$.

We may assume that of the irreducible factors* of $\Phi$ there are none of degree lower than $m$ vanishing at the point (0, 0, 0), for otherwise each of such factors could be treated separately by the methods here used, and the results combined. This provision excludes the case in which one of the variables has equal roots for all values of the other two in the neighborhood of the point (0, 0, 0).

4. The quadratic transformation

$$\xi = \bar{\xi}\,\zeta \qquad\qquad \eta = \eta \tag{2}$$

reduces $\Phi(\xi, \eta, \zeta)$ to the form

$$\begin{aligned}\Phi(\xi, \eta, \zeta) &= \zeta^m [\phi(\bar{\xi}, \bar{\eta}) + \zeta\,\chi(\bar{\xi}, \bar{\eta}, \zeta)] \\ &= \xi^m \bar{\Phi}(\bar{\xi}, \bar{\eta}, \zeta)\end{aligned} \tag{3}$$

where, an arbitrarily large positive number $\Gamma$ having been chosen at pleasure, $\delta$ can be so determined that the function $\bar{\Phi}$ will be analytic when

$$|\bar{\xi}| < \Gamma, \qquad |\bar{\eta}| < \Gamma, \qquad |\zeta| < \delta.$$

Equation (3) follows at once from the intermediate form

$$\Phi(\xi, \eta, \zeta) = \zeta^m [(\bar{\xi}, \bar{\eta}, 1)_m + \zeta(\bar{\xi}, \bar{\eta}, 1)_{m+1} + \zeta^2(\bar{\xi}, \bar{\eta}, 1)_{m+2} + \ldots\ldots].$$

We now proceed to the proof that the function $\bar{\Phi}(\bar{\xi}, \bar{\eta}, \zeta)$ is analytic within the above limits.

$$\text{Let } \Phi(\xi, \eta, \zeta) = \Sigma A_{ijk}\,\xi^i \eta^j \zeta^k, \qquad i + j + k \geqq m,$$

and suppose it to be convergent when

$$|\xi| < h, \qquad |\eta| < h, \qquad |\zeta| < h \qquad h > \delta_1.$$

Then, for the general term, we have

$$|A_{ijk}|\,\delta_1^{i+j+k} < M,$$

$M$ being a positive constant.

By transformation (2)

$$\Phi(\xi, \eta, \zeta) = \Sigma A_{ijk}\,\bar{\xi}^i \bar{\eta}^j \zeta^{i+j+k}$$

* For the definition and the fundamental properties of the irreducible factors of an analytic function of several variables, which vanishes in a point, cf. Encyclopädie der mathematischen Wissenschaften, II. B. 1, Nr. 45.

and

$$\bar{\Phi}(\bar{\xi}, \bar{\eta}, \zeta) = \Sigma A_{ijk} \bar{\xi}^i \bar{\eta}^j \zeta^{i+j+k-m}$$

Now choose $\delta$ so that $\Gamma \delta \leqq \delta_1$.

Then, when

$$|\bar{\xi}| = \Gamma_1, \qquad |\bar{\eta}| = \Gamma_1, \qquad |\zeta| = \delta, \qquad \Gamma_1 > 1,$$

the absolute value of the general term of series $\bar{\Phi}$ becomes

$$\leqq |A_{ijk}| \cdot \Gamma_1^{i+j} \cdot \delta^{i+j+k-m}$$

$$\leqq |A_{ijk}| \cdot \delta_1^{i+j} \cdot \delta^{k-m}$$

$$< \frac{|A_{ijk}| \cdot \delta_1^{i+j+k}}{\delta^m}$$

$$< \frac{M}{\delta^m}$$

Accordingly, the series

$$\Sigma A_{ijk} \bar{\xi}^i \bar{\eta}^j \zeta^{i+j+k-m}$$

is convergent when

$$|\bar{\xi}| < \Gamma_1 \qquad |\bar{\eta}| < \Gamma_1 \qquad |\zeta| < \delta,^*$$

and it represents an analytic function for these values of the arguments.

5. The family of lines tangent to the surface $\Phi(\xi, \eta, \zeta) = 0$ at the point $(0, 0, 0)$ forms a cone that cuts the plane $\zeta = 1$ in the curve $\phi(\bar{\xi}, \bar{\eta}) = 0$. If the line $\eta/\zeta = \beta, \xi/\zeta = \alpha$, ($\alpha$ and $\beta$ being finite) is one of this family, then the point $\bar{\xi} = \alpha, \bar{\eta} = \beta, \zeta = 0$ of the surface $\bar{\Phi}(\bar{\xi}, \bar{\eta}, \zeta) = 0$, (3) is at most a singular point of order $m$ of that surface, and its neighborhood corresponds to a portion of the neighborhood of the singular point of the original surface $\Phi(\xi, \eta, \zeta) = 0$. In fact, cut the surface

$$\Phi(\xi, \eta, \zeta) = 0$$

by the plane

$$\eta - \beta\zeta = 0.$$

Then the curve of intersection $C$ will have a multiple point at $(0, 0, 0)$ and the equations of the tangents to $C$ at $(0, 0, 0)$ will be

$$\left.\begin{aligned} \eta - \beta\zeta &= 0 \\ \xi - \alpha_\sigma\zeta &= 0 \end{aligned}\right\} \qquad \sigma = 1, 2, \ldots\ldots s \leqq m.$$

* Cf. Stolz, Allgemeine Arithmetik, Vol. I. p. 293.

Now, the transformation (2) being made, the points of the region

$$T: \quad |\xi| < \delta, \qquad |\eta| < \delta, \qquad |\zeta| < \delta,$$

which lie in the neighborhoods of the lines

$$\left.\begin{aligned} \eta - \beta\zeta &= 0 \\ \xi - a_\sigma\zeta &= 0 \end{aligned}\right\} \qquad \sigma = 1, 2, \ldots\ldots s,$$

can, with the exception of the point (0, 0, 0), be transformed in a one-to-one manner on the neighborhoods of the points (0, 0, 0) of a set of surfaces

$$g_\sigma(\xi_\sigma, \eta_1, \zeta) = 0, \qquad \sigma = 1, 2, \ldots\ldots s,$$

the coördinates being connected by the relation

$$\left.\begin{aligned} \xi &= \zeta(\xi_\sigma + a_\sigma) \\ \eta &= \zeta(\eta_1 + \beta) \end{aligned}\right\} \tag{4}$$

By the *neighborhood* of the above line is here meant the set of points $(\xi, \eta, \zeta)$ which satisfy the condition

$$|\xi - a_\sigma\zeta| \leqq \epsilon|\zeta|, \qquad |\eta - \beta\zeta| \leqq \epsilon|\zeta|, \qquad |\zeta| < \delta.$$

To deal with the points for which $\alpha$, $\beta$ would be infinite, cut the surface

$$\Phi(\xi, \eta, \zeta) = 0$$

by the plane

$$\zeta = 0.$$

The equations of the tangents to the curve of intersection at (0, 0, 0) are

$$\left.\begin{aligned} \xi - a_\tau\eta &= 0 \\ \zeta &= 0 \end{aligned}\right\} \qquad \tau = 1, 2, \ldots\ldots t \leqq m.$$

By means of a transformation corresponding to (2),

$$\xi = \bar{\xi}\eta, \qquad \zeta = \bar{\zeta}\eta,$$

the points of T which lie in the neighborhoods of the lines

$$\left.\begin{aligned} \xi - a_\tau\eta &= 0 \\ \zeta &= 0 \end{aligned}\right\} \qquad \tau = 1, 2, \ldots\ldots t \leqq m,$$

can, with the exception of the point (0, 0, 0), be transformed in a one-to-one manner on the neighborhoods of the points (0, 0, 0) of the set of surfaces

$$g_\tau(\bar{\xi}_\tau, \eta, \bar{\zeta}) = 0, \qquad \tau = 1, 2, \ldots\ldots t \leqq m,$$

the coördinates being connected by the relations

$$\left.\begin{aligned} \xi &= \eta\,(\bar{\xi}_\tau + a_\tau) \\ \zeta &= \eta\,\bar{\bar{\zeta}} \end{aligned}\right\} \tag{5}$$

By the neighborhood of the line

$$\left.\begin{aligned} \xi - a_\tau\,\eta &= 0 \\ \zeta &= 0 \end{aligned}\right\}$$

is here meant the set of points $(\xi, \eta, \zeta)$ which satisfy the condition

$$|\xi - a_\tau\,\eta| \leqq \epsilon\,|\eta|, \qquad |\zeta| \leqq \epsilon\,|\eta|, \qquad |\eta| < \delta.$$

The singularities of the surfaces

$$g_\sigma\,(\xi_\sigma, \eta_1, \zeta) = 0,$$
$$g_\tau\,(\bar{\xi}_\tau, \eta, \bar{\bar{\zeta}}) = 0$$

at the points $(0, 0, 0)$ are at most of order $m$. Their further properties will be considered later.

Let $G$ be an arbitrarily chosen (large) positive quantity, $\delta$ a second suitably chosen positive quantity: then any point of $T$, for which

$$|\xi| < G\,|\zeta|, \qquad |\eta| < G\,|\zeta|, \qquad 0 < |\zeta| < \delta,$$

is carried by the transformation (4) into one of the neighborhoods above considered on the surfaces $g_\sigma = 0$. If $\zeta = 0$, but $\xi$, $\eta$ do not both vanish, then the point $(\xi, \eta, \zeta)$ is carried by (5) into one of the neighborhoods considered on the surfaces $g_\tau = 0$.

In (3), the function $\phi\,(\bar{\xi}, \bar{\eta})$ contains the term $\bar{\xi}^m$ by 3, 2). Apply the transformation

$$\bar{\eta} - \beta = \eta_1, \tag{6}$$

whence (3) takes the form

$$\Phi\,(\xi, \eta, \zeta) = \zeta^m\,[\phi_1\,(\bar{\xi}, \eta_1) + \zeta\,\chi_1\,(\bar{\xi}, \eta_1, \zeta)].$$

In $\phi_1\,(\bar{\xi}, \eta_1)$, take out all terms not containing $\eta_1$, so that

$$\phi_1\,(\bar{\xi}, \eta_1) = \prod_{\sigma=1}^{s} (\bar{\xi} - a_\sigma)^{\mu_\sigma} + \eta_1\,\psi_1\,(\bar{\xi}, \eta_1), \quad \mu_1 + \ldots\ldots + \mu_s = m.$$

Then make the transformation

$$\bar{\xi} - a_\sigma = \xi_\sigma \tag{6'}$$

and we have

$$\Phi\,(\xi, \eta, \zeta) = \zeta^m\,[\xi_\sigma^{\mu_\sigma} \prod_{\sigma'=1}^{s}{}' (\xi_\sigma + a_\sigma - a_{\sigma'})^{\mu_{\sigma'}} + \eta_1\,\psi\,(\xi_\sigma, \eta_1) + \zeta\,\chi\,(\xi_\sigma, \eta_1, \zeta)]$$

$$= \zeta^m g_\sigma\,(\xi_\sigma, \eta_1, \zeta) = 0 \tag{7}$$

where $g_\sigma$ has a term in $\xi_\sigma^{\mu_\sigma}$ free from $\eta_1$ and $\zeta$, since $a_\sigma - a_{\sigma'} \neq 0$. So there are near the point (0, 0, 0) $\mu_\sigma$ values of $\xi_\sigma$ satisfying the equation $g_\sigma = 0$ for every pair of values of $\eta_1$ and $\zeta$ in the neighborhood of the point $\eta_1 = 0$, $\zeta = 0$. Now, for any such set of values of $\xi_\sigma$, $\eta_1$, $\zeta$, different from the set (0, 0, 0), satisfying the equation $g_\sigma = 0$, there is a corresponding set of values of $\xi$, $\eta$, $\zeta$ satisfying the equation $\Phi(\xi, \eta, \zeta) = 0$, their coördinates being connected by the relations (2), (6), and (6′), which are equivalent to the required relation (4). Also by considering the other factors of $\prod_{\sigma=1}^{s} (\bar{\xi} - a_\sigma)^{\mu_\sigma}$, we get $(s-1)$ other equations of form (7), the corresponding coördinates being connected by relations of form (4).

No two points $(\xi, \eta, \zeta)$, $(\xi', \eta', \zeta')$ of $T$ (distinct from (0, 0, 0)), derived from points $(\xi_\sigma, \eta_1, \zeta_1)$ $(\xi_{\sigma'}, \eta_2, \zeta_2)$ lying respectively in the neighborhoods of the singularities which are given by two distinct equations

$$g_\sigma = 0, \qquad g_{\sigma'} = 0,$$

can be the same. For suppose

$$\begin{aligned} \xi = \xi' &= \zeta_1 (\xi_\sigma + a_\sigma) = \zeta_2 (\xi_{\sigma'} + a_{\sigma'}) \\ \eta = \eta' &= \zeta_1 (\eta_1 + \beta) = \zeta_2 (\eta_2 + \beta) \\ \zeta = \zeta' &= \zeta_1 \qquad\qquad = \zeta_2 \end{aligned}$$

Then we must have

$$\begin{aligned} \xi_\sigma + a_\sigma &= \xi_{\sigma'} + a_{\sigma'}, \\ \xi_\sigma - \xi_{\sigma'} &= a_{\sigma'} - a_\sigma, \end{aligned}$$

and, by taking the neighborhoods of the singularities in question sufficiently small, we can insure that the difference $\xi_\sigma - \xi_{\sigma'}$ is less in absolute value than the difference $a_{\sigma'} - a_\sigma$. In a similar manner it is shown that, if the equation $g_\sigma = 0$, regarded as an equation in $\xi_\sigma$, has equal roots for all values of $\eta_1$, $\zeta$ in the neighborhood of the point $\eta_1 = 0$, $\zeta = 0$, the equation $\Phi = 0$ must also have equal roots at the corresponding points, and this case has been excluded. So as each equation $g_\sigma = 0$ has near the point (0, 0, 0) $\mu_\sigma$ values of $\xi_\sigma$, in general distinct, for each pair of values of $\eta_1$ and $\zeta$, and as $\sum_{\sigma=1}^{s} \mu_\sigma = m$, the collection of equations

$$g_\sigma = 0, \qquad \sigma = 1, 2, \ldots\ldots s,$$

has within sufficiently small limits as many different roots as the equa-

tion $\Phi = 0$, and thus represents the latter equation within the corresponding limits, i. e., when

$$|\zeta| < \delta, \qquad |\eta_1| < \epsilon, \qquad \xi_\sigma| < \epsilon,$$

or

$$|\xi - a_\sigma \zeta| < \epsilon |\zeta|, \qquad |\eta - \beta\zeta| < \epsilon |\zeta|, \qquad |\zeta| < \delta.$$

Next we consider points for which $\zeta = 0$, but $\xi$, $\eta$ are not both zero. For these we use the transformation

$$\xi = \bar{\bar{\xi}}\,\eta, \qquad \zeta = \bar{\bar{\zeta}}\,\eta. \tag{8}$$

Then, by the same method of treatment as above, putting $\bar{\bar{\zeta}}$ for $\bar{\eta}$ and $\eta$ for $\zeta$, and taking $\beta = 0$, we derive a set of surfaces

$$g_\tau(\xi_\tau, \eta, \bar{\bar{\zeta}}) = 0, \qquad \tau = 1, 2, \ldots\ldots t \leqq m,$$

on which are mapped all points of the original neighborhood for which

$$|\eta| < \delta_1, \qquad \left|\frac{\zeta}{\eta}\right| < \epsilon_1,$$

and so all points for which

$$|\eta| < \delta_1, \qquad |\zeta| < \epsilon_1 |\eta|, \qquad |\xi - a_\tau \eta| < \epsilon_1 |\eta|.$$

Here, we have a function corresponding to $\phi(\bar{\xi}, \bar{\eta})$:

$$\bar{\phi}(\bar{\bar{\xi}}, \bar{\bar{\zeta}}) = (\bar{\bar{\xi}}, 1, \bar{\bar{\zeta}})_m$$

Now, for the infinite roots of

$$\phi(\bar{\xi}, \bar{\eta}) = 0,$$

we put the equation into the form

$$\left(\frac{\bar{\xi}}{\bar{\eta}}, 1, \frac{0}{\bar{\eta}}\right)_m = 0.$$

So the equation

$$\bar{\phi}(\bar{\bar{\xi}}, \bar{\bar{\zeta}}) = 0$$

is such that its roots for $\bar{\bar{\xi}} = 0$ are the same as the ratios of the infinite roots of the equation

$$\phi(\bar{\xi}, \bar{\eta}) = 0,$$

and by 3, 3) these ratios are all finite.

## C. — The Number of the Neighborhoods, $t_1, t_2, \ldots\ldots t_\nu$, required to represent $T$ is Finite.

6. In the foregoing paragraph it has been shown that the neighborhood of each tangent line to the surface $\Phi = 0$, at the singular point can be mapped on the neighborhood of a (regular or singular) point of the surface $g_\sigma = 0$. We now proceed to show that the whole neighborhood

$$T: \qquad |\xi| < \delta, \qquad |\eta| < \delta, \qquad |\xi| < \delta$$

can be covered by the neighborhoods of a finite number of such lines. We distinguish two cases: —

Case I. — The polynomial $\phi(\bar{\xi}, \bar{\eta})$ has no multiple factors.

Case II. — This polynomial has multiple factors.

Theorem: *The neighborhood $T$ can be completely covered by a finite number of regions $T_1, T_2, \ldots\ldots T_\nu$, which overlap each other and which are mapped respectively on the following regions $t_1, t_2, \ldots\ldots t_\nu$:*

*In Case* I: 1) *the region $t_i$, $i = 1, 2, \ldots\ldots \kappa$, consists of the neighborhood of a singular point of the surface $g^{(i)} = 0$;*

2) *the extent of each of the neighborhoods $t_1, t_2, \ldots\ldots t_\kappa$ having been arbitrarily determined, the regions $t_j$, $j = \kappa + 1, \ldots\ldots \nu$, then consist of regular regions of surfaces $g^{(j)} = 0$.*

*In Case* II: 1) *the region $t_i$, $i = 1, 2, \ldots\ldots \kappa$, consists of the neighborhood of a singular point of the surface $g^{(i)} = 0$;*

2) *the extent of each of the neighborhoods $t_1, t_2, \ldots\ldots t_\kappa$ having been arbitrarily determined, the regions $t_j, j = \kappa + 1, \ldots\ldots \nu$, then consist of regions of surfaces $g^{(j)} = 0$ defined as follows: omitting the index $j$ throughout, we write*

$$g(\xi_\sigma, \eta_1, \zeta) = [\xi_\sigma^r + p_1(\eta_1, \zeta)\,\xi_\sigma^{r-1} + \ldots\ldots + p_r(\eta_1, \zeta)]\,E(\xi_\sigma, \eta_1, \zeta),$$

*where $p_e(\eta_1, \zeta)$ is analytic throughout a region*

$$|\eta_1| < h, \qquad |\xi| < \delta.$$

*Here $r$, for a given value of $j$, is a positive integer satisfying the relation $1 \leqq r \leqq m$.*

Case I. — The polynomial $\phi(\bar{\xi}, \bar{\eta})$ contains no multiple factors.

Here, the equation

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$

can have multiple values of $\bar{\xi}$ only for a finite number of values of $\bar{\eta}$, these being the values for which the equations

$$\phi = 0, \qquad \frac{\partial \phi}{\partial \bar{\xi}} = 0$$

have common roots, and by the condition 3, 3) none of these values of $\bar{\eta}$ become infinite.

Now we consider all such values of $\bar{\eta}$

$$\bar{\eta} = c_r, \qquad r = 1, 2, \ldots\ldots l,$$

for which the equation, considered as an equation in $\bar{\xi}$,

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$

has multiple roots. Deal with each of these as in 5, $c_r$ taking the place of $\beta$ in (6); then, in equation (7), some of the $\mu_\sigma$'s will, in general, be greater than unity, i. e. some of the equations $g_\sigma = 0$ will have for the lowest terms in $\xi_\sigma$ alone exponents greater than 1. For such as have their $\mu_\sigma = 1$, there are regular points. The others will afford singular points unless they have terms of the first degree in either $\eta_1$ or $\zeta$. Surround these points by neighborhoods

$$|\xi_\sigma| < \bar{\delta}, \qquad |\eta_1| < \bar{\delta}, \qquad |\zeta| < \bar{\delta},$$

i. e.

$$|\bar{\xi} - a_\sigma| < \bar{\delta}, \qquad |\bar{\eta} - c_r| < \bar{\delta}, \qquad |\zeta| < \bar{\delta},$$

which are to be considered later.

Now let $\bar{\eta} = b$ be any value for which the equation

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$

has not equal roots. Then the equations $g_\sigma = 0$ of (7) each have a term in $\xi_\sigma$ to the first degree, free from $\eta_1$ and $\zeta$, and thus the points of the surface $g_\sigma = 0$ lying in the neighborhood of the point $\xi_\sigma = 0$, $\eta_1 = 0$, $\zeta = 0$, can be represented by a power series

$$\xi_\sigma = p(\eta_1, \zeta).$$

So, in this case, we have $m$ developments

$$\xi_\sigma = \bar{p}_\sigma(\eta_1, \zeta), \qquad \sigma = 1, 2, \ldots\ldots m,$$

and, by using the relations (4), we have

$$\xi = p_\sigma(\eta, \zeta), \qquad \sigma = 1, 2, \ldots\ldots m.$$

It is readily seen that the function

$$\xi = p(\eta_1, \zeta)$$

is analytic within the region

$$|\eta_1| < h - e, \qquad |\zeta| < \delta_1 \neq 0,$$

where $h$ is the distance to the nearest value of $\eta_1$ for which the equation corresponding to

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$

has a critical point, $e$ is a positive number which can be taken arbitrarily small and, having been chosen, determines an upper limit, not zero, for $\delta_1$. In fact, $\xi_\sigma$ is a continuous function of the two independent variables $\eta_1$, $\zeta$ within this region; furthermore, for any fixed value of $\zeta$ such that $|\zeta| < \delta_1$, $\xi_\sigma$ is an analytic function of $\eta_1$ throughout the region $|\eta_1| < h - e$; and, similarly, for any fixed value of $\eta_1$ such that $|\eta_1| < h - e$, $\xi_\sigma$ is an analytic function of $\zeta$ throughout the region $|\zeta| < \delta_1$.*

Also consider the surfaces

$$g_\tau(\xi_\tau, \eta, \bar{\bar{\zeta}}) = 0$$

in 5. Here also we have $m$ regular points of surfaces, and as a result $m$ functions of the form

$$\xi_\tau = q(\eta, \bar{\bar{\zeta}}).$$

These, by the same method of proof as above, are seen to be analytic when

$$|\bar{\bar{\zeta}}| < h_1 - e_1, \qquad |\eta| < \delta_2,$$

where $h$ is the nearest point in the $\bar{\bar{\zeta}}$-plane for which the equation

$$\bar{\phi}(\bar{\bar{\xi}}, \bar{\bar{\zeta}}) = 0$$

has multiple roots for $\xi$, i. e. the smallest value of $\bar{\bar{\zeta}}$ for which the equation

$$(\bar{\bar{\xi}}, 1, \bar{\bar{\zeta}})_m = 0$$

has equal roots for $\bar{\bar{\xi}}$. But this is the smallest value of $\frac{1}{\eta}$ for which the equation

$$\left(\frac{\xi}{\eta}, 1, \frac{1}{\eta}\right)_m = 0$$

* Cf. Briot et Bouquet's Théorie des fonctions elliptiques, § 28. The proof of continuity there given for polynomials in two variables will apply with very slight modifications to analytic functions of any number of variables. Cf. further Jordan's Cours d'analyse, I. § 206, § 258.

regarded as an equation in $\xi/\eta$, has equal roots. Thus $\frac{1}{h_1}$ is the largest value of $\bar{\eta}$ for which the equation

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$

has a critical point. So the functions are analytic and give all points of the original neighborhood for which

$$|\bar{\zeta}| = \left|\frac{\zeta}{\eta}\right| < h_1 - e_1, \qquad |\eta| < \delta_2,$$

or for which

$$\left|\frac{\eta}{\zeta}\right| > \frac{1}{h_1 - e_1} = \frac{1}{h_1} + \frac{e_1}{h_1(h_1 - e_1)} = h_2 + e_2, \quad \left(h_2 = \frac{1}{h_1}\right),$$

$$|\zeta| = |\bar{\zeta}|\,|\eta| < \delta_2(h_1 - e_1) = \delta_3,$$

thus securing the limits

$$|\eta| < \delta_2, \qquad |\zeta| < \delta_3, \qquad |\eta| > (h_2 + e_2)|\zeta|,$$

where $h_2$ is the distance to the furthest point in the $\bar{\eta}$-plane for which the equation

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$

has a critical point, and if $e_2$ is first chosen arbitrarily small, $\delta_3$ can be determined not zero.

Now consider the neighborhoods of the critical points of the curve

$$\phi(\bar{\xi}, \bar{\eta}) = 0.$$

In these, however small we take the $\bar{\delta}$, all the remainder of a circle in the $\bar{\eta}$-plane including all the values for which the curve $\phi = 0$ has critical points can be covered with circles such as were determined for the domains of the regular points above, these circles overlapping the circles about the singular points and not reaching out to these points in any case. Let the radius of the large circle be $G$ where

$$G > 1, \qquad G \geqq h_2 + e_2.$$

Then, if we take for $\delta_4$ the smallest value of any $\delta_1$ or $\bar{\delta}$, the developments within these circles together with the neighborhoods of the set of new singular points will represent all points of the original neighborhood for which

$$|\eta| < G\delta_4, \qquad |\zeta| < \delta_4.$$

Finally, taking for $\delta$ the smallest of the three quantities $\delta_2$, $\delta_3$, $\delta_4$, the

whole set of functions thus determined will represent all points of the original neighborhood for which

$$|\eta| < \delta, \qquad |\zeta| < \delta.$$

The new set of singular points may or may not be all of degrees lower than $m$, but if they are we have simplified the problem; we have *reduction*, as we shall say, borrowing a term frequently used in the theory of algebraic invariants of a linear transformation; and if not, the further treatment will be considered later.

### D. — An Example.

Before taking up Case II, however, we consider an example in which the degree is reduced by one quadratic transformation, and the parametric representation $(A)$ is at once secured.

Let the surface be

$$\Phi(\xi, \eta, \zeta) = \xi^2 + \eta^2 - \zeta^2 - \xi\zeta^2 = 0.$$

The transformation

$$\xi = \bar{\xi}\zeta, \qquad \eta = \bar{\eta}\zeta,$$

secures for the equation corresponding to (3)

$$\bar{\Phi}(\bar{\xi}, \bar{\eta}, \zeta) = \bar{\xi}^2 + \bar{\eta}^2 - 1 - \bar{\xi}\zeta = 0.$$

Here

$$\phi(\bar{\xi}, \bar{\eta}) = \bar{\xi}^2 + \bar{\eta}^2 - 1$$

and the critical points are

$$\bar{\xi} = 0, \qquad \bar{\eta} = 1,$$
$$\bar{\xi} = 0, \qquad \bar{\eta} = -1.$$

Let

$$\xi_1 = \bar{\xi}, \qquad \eta_1 = \bar{\eta} - 1,$$

and we have

$$\xi_1^2 + \eta_1^2 + 2\eta_1 - \xi_1\zeta = 0.$$

Hence

$$\eta_1 = -1 + \sqrt{\xi_1(\zeta - \xi_1) + 1}. \qquad \text{(a)}$$

Also let

$$\xi_2 = \bar{\xi}, \qquad \eta_2 = \bar{\eta} + 1,$$

and we have

$$\eta_2 = 1 - \sqrt{\xi_2(\zeta - \xi_2) + 1}. \qquad \text{(b)}$$

In (a) and (b), only that branch of the radical is taken which becomes + 1 for zero values of the arguments.

Again, we make the transformation

$$\xi = \bar{\xi}\eta, \qquad \zeta = \bar{\zeta}\eta,$$

and derive the surface

$$\bar{\xi}^2 + 1 - \bar{\zeta}^2 - \bar{\xi}\bar{\zeta}^2\eta = 0.$$

Here

$$\bar{\phi}(\bar{\xi}, \bar{\zeta}) = \bar{\xi}^2 + 1 - \bar{\zeta}^2$$

and for the value $\bar{\zeta} = 0$ we have the roots

$$\bar{\xi} = i, \qquad \bar{\xi} = -i.$$

Let

$$\xi_3 = \bar{\xi} - i,$$

and we have the surface

$$\xi_3^2 + 2i\xi_3 - \bar{\zeta}^2 - \bar{\zeta}^2\eta\xi_3 - i\bar{\zeta}^2\eta = 0,$$

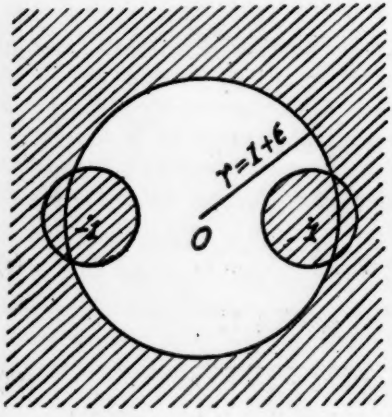
whence

$$\xi_3 = \frac{\bar{\zeta}^2\eta - 2i}{2} + \tfrac{1}{2}\sqrt{\bar{\zeta}^4\eta^2 + 4\bar{\zeta}^2 - 4}. \qquad \text{(c)}$$

In a similar way, from the other root,

$$\xi_4 = \frac{\bar{\zeta}^2\eta + 2i}{2} - \tfrac{1}{2}\sqrt{\bar{\zeta}^4\eta^2 + 4\bar{\zeta}^2 - 4}. \qquad \text{(d)}$$

In (c) and (d), for the radical is taken only that branch which becomes $+2i$ for zero values of the arguments, and the function is seen to be analytic for sufficiently small values of $\eta$ when




$$|\bar{\zeta}| = \left|\frac{\xi}{\eta}\right| < 1 - \epsilon_1;$$

and similarly when

$$|\bar{\eta}| = \left|\frac{\eta}{\xi}\right| > 1 + \epsilon.$$

Thus, in the $\bar{\eta}$-plane, we have by the formulas (a), (b), (c), (d) covered two small circles about the points 1 and $-1$ corresponding to developments (a) and (b), and all of the region outside of a circle of radius $(1 + \epsilon)$, corresponding to developments (c) and (d). We must now obtain further formulas so as to fill up the remaining unshaded region.

Consider the point

$$\bar{\xi} = -1, \qquad \bar{\eta} = 0.$$

Let $\bar{\xi} = \xi_5 - 1$ and we have

$$\xi_5^2 - 2\xi_5 + \bar{\eta}^2 - \xi_5\zeta + \zeta = 0,$$

whence

$$\xi_5 = \frac{\zeta + 2}{2} - \tfrac{1}{2}\sqrt{\zeta^2 - 4\bar{\eta}^2 + 4}. \tag{e}$$

In the same way, about the point

$$\bar{\xi} = 1, \qquad \bar{\eta} = 0,$$

we have the function

$$\xi_6 = \frac{\zeta - 2}{2} + \tfrac{1}{2}\sqrt{\zeta^2 - 4\bar{\eta}^2 + 4} \tag{f}$$

In (e) and (f), for the radical we take only that branch which becomes $+ 2$ for zero values of the argument, and for sufficiently small values of $\xi$ the functions are analytic when

$$|\bar{\eta}| < 1 - \epsilon_2.$$

Again, consider the point

$$\bar{\xi} = 2\sqrt{1-i}, \qquad \bar{\eta} = 1 + 2i.$$

Let

$$\bar{\xi} = \xi_7 + 2\sqrt{1-i}, \qquad \bar{\eta} = \eta_7 + 1 + 2i,$$

and we have

$$\xi_7^2 + 4\sqrt{1-i}\,\xi_7 + \eta_7^2 + 2(1+2i)\eta_7 - \xi_7\zeta - 2\sqrt{1-i}\,\zeta = 0,$$

whence

$$\xi_7 = -\frac{4\sqrt{1-i} - \zeta}{2} + \tfrac{1}{2}\sqrt{16 - 16i + \zeta^2 - 4\eta_7^2 - 8(1+2i)\eta_7}. \tag{g}$$

For the corresponding point

$$\bar{\xi} = -2\sqrt{1-i}, \qquad \bar{\eta} = 1 + 2i,$$

we have the formula

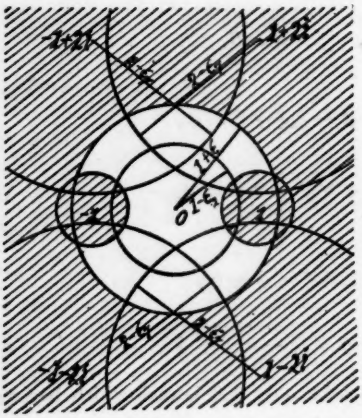
$$\xi_8 = \frac{4\sqrt{1-i} + \zeta}{2} - \tfrac{1}{2}\sqrt{16 - 16i + \zeta^2 - 4\eta_8^2 - 8(1+2i)\eta_8}. \tag{h}$$

In (g) and (h), for the radical we take only the branch which becomes $+ 4\sqrt{1-i}$ for zero values of the arguments, the same value of the radical $\sqrt{1-i}$ being taken in all cases. These functions are analytic for sufficiently small values of $\zeta$ when

$$|\eta_7| = |\eta_8| < 2 - \epsilon_7.$$

Also, considering the corresponding points of

$$\phi(\bar{\xi}, \bar{\eta}) = 0$$




for which

$$\bar{\eta} = -1 + 2i,$$
$$\bar{\eta} = \quad 1 - 2i,$$
$$\bar{\eta} = -1 - 2i,$$

we have evidently similar regions for each. Then, by taking the $\epsilon$'s all small enough, we cover the whole unshaded remainder of the $\bar{\eta}$-plane by five circles within each of which there is a development as required.

The sets of parametric formulae, derived by using the intermediate transformations, are

$$\begin{cases} \xi = uv, & \eta = \quad v\sqrt{u(v-u)+1}, & \zeta = v \quad \text{from} \quad \text{(a)} \\ \xi = uv, & \eta = -v\sqrt{u(v-u)+1}, & \zeta = v \quad " \quad \text{(b)} \end{cases}$$

$$\begin{cases} \xi = \frac{v}{2}(u^2 v + \sqrt{u^4 v^2 + 4u^2 - 4}), & \eta = v, \quad \zeta = uv \quad " \quad \text{(c)} \\ \xi = \frac{v}{2}(u^2 v - \sqrt{u^4 v^2 + 4u^2 - 4}), & \eta = v, \quad \zeta = uv \quad " \quad \text{(d)} \end{cases}$$

$$\begin{cases} \xi = \frac{u}{2}(u - \sqrt{u^2 - 4v^2 + 4}), & \eta = uv, \quad \zeta = u \quad " \quad \text{(e)} \\ \xi = \frac{u}{2}(u + \sqrt{u^2 - 4v^2 + 4}), & \eta = uv, \quad \zeta = u \quad " \quad \text{(f)} \end{cases}$$

$$\begin{cases} \xi = \frac{u}{2}\left( u + \sqrt{16 - 16i + u^2 - 4v^2 - 8(1+2i)v} \right), \\ \qquad \eta = u(v + 1 + 2i), \qquad \zeta = u \quad " \quad \text{(g)} \\ \xi = \frac{u}{2}\left( u - \sqrt{16 - 16i + u^2 - 4v^2 - 8(1+2i)v} \right), \\ \qquad \eta = u(v + 1 + 2i), \qquad \zeta = u \quad " \quad \text{(h)} \end{cases}$$

with three more sets similar to (g) and (h).

Case II. — The polynomial $\phi(\bar{\xi}, \bar{\eta})$ contains multiple factors.

Here, any points which are common to two different irreducible factors of $\phi(\bar{\xi}, \bar{\eta})$, or are critical points of a single irreducible factor, will be critical points of the curve

$$\phi(\bar{\xi}, \bar{\eta}) = 0,$$

and all such points, a finite number in all, will be treated like the critical points of the previous case. But also any point on a multiple factor is a critical point of the curve, and further treatment is needed for such points.

Suppose $\bar{\xi} = a$, $\bar{\eta} = \beta$ is a regular point of a factor $\phi_1(\bar{\xi}, \bar{\eta})$ of multiplicity $r$, i. e. of the irreducible factor whose $r$th power is equal to $\phi_1(\bar{\xi}, \bar{\eta})$ and not a point of any different factor of $\phi(\bar{\xi}, \bar{\eta})$. Then, in the corresponding equation of form (7), $g_\sigma$ will contain a term $\xi_\sigma^r$ as the lowest term in $\xi_\sigma^r$ free from $\eta_1$ and $\zeta$, and by Weierstrass's Theorem * we can develop the function about the point in the form

$$\left[\xi_\sigma^r + p_1(\eta_1, \zeta)\xi_\sigma^{r-1} + \cdots\cdots + p_r(\eta_1, \zeta)\right] E(\xi_\sigma, \eta_1, \zeta) = 0. \quad (9)$$

These functions

$$p_\lambda(\eta_1, \zeta), \qquad \lambda = 1, 2, \ldots\ldots r,$$

are shown by a method similar to that used for the functions in Case I to be analytic within a region

$$|\eta_1| < h - e, \qquad |\zeta| < \delta_1,$$

where $h$ is the distance to the nearest value of $\eta_1$ which gives a point of intersection of two different irreducible curves corresponding to factors of $\phi(\bar{\xi}, \bar{\eta})$, or to a critical point of one of the irreducible curves.

Now none of the excepted points can be at infinity, on account of the provision in 3, 3). So the points on the surfaces $g_\tau = 0$ in 5 will also afford developments of order (9), and by the method of Case I, we have a similar region for the convergence of the coefficients of the different powers of $\xi_\tau$ in the polynomial, i. e. the exterior of a circle including all of the excepted points.

Accordingly, in this case also, we represent the neighborhood of the original singular point by a finite number of neighborhoods of new singular points together with a finite number of functions, some of which are now not analytic for the values of the arguments considered, but satisfy equations of the form

$$\xi_\sigma^r + p_1(\eta_1, \zeta)\xi_\sigma^{r-1} + \cdots\cdots + p_r(\eta_1, \xi) = 0. \quad (11)$$

For the further treatment of these functions, we shall establish an auxiliary theorem in § 2.

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* See Picard's Traité d'analyse, Vol. II. p. 241.

Any point in $T$ can be carried by a suitable transformation into a point on one of the surfaces $g_\sigma = 0$ or $g_\tau = 0$. Let $G$ be an arbitrarily chosen (large) positive quantity; then any point of $T$ for which

$$|\xi| < G|\zeta|, \qquad |\eta| < G|\zeta|, \qquad |\zeta| < \delta,$$

is carried by the transformation (4) into one of the neighborhoods considered on the surfaces $g_\sigma = 0$.

If $\zeta = 0$, but $\xi$, $\eta$ do not both vanish, then the point $(\xi, \eta, s)$ is carried by (5) into one of the neighborhoods considered on the surfaces $g_\tau = 0$.

## § 2.

### A. — A Lemma.

1. The treatment of the multiple curves of Case II depends on the following

Lemma. — *Given an analytic surface*

$$\Phi(x, y, z) = \phi(x, y) + z\Psi(x, y, z) = 0, \qquad (\alpha)$$

*where*

$$\phi(x, y) = 0$$

*is a multiple curve; let $\phi(x, y)$ have the form in the neighborhood of the point $x = 0, y = 0$,*

$$\phi(x, y) = [x + p(y)]^m E(x, y), \qquad (\beta)$$

*where $p(y)$ is analytic at the point $y = 0$, and $p(0) = 0$. The function $\Psi(x, y, z)$ shall be analytic at the point $(0, 0, 0)$, but shall not be divisible by $x + p(y)$ at that point. Consider a region for which $|y| < h$, and let $h$ be chosen*

*a) less than the radius of convergence of the Taylor's series which represents the function $p(y)$ developed about the point $y = 0$, and*

*b) sufficiently small, so that the points $(x = p(y), y)$ will lie in the region in which $E(x, y)$ is analytic and different from zero. Then the part of the neighborhood of the curve*

$$x + p(y) = 0, \qquad z = 0,$$

*which lies on the surface*

$$\Phi(x, y, z) = 0$$

*can be transformed, by means of quadratic transformations of the type*

$$x = \bar{x}z,$$

*on a finite number of regions* $\tau_1, \tau_2, \ldots\ldots \tau_\nu$, *which fall into two categories:* —

1) *the region* $\tau_i\,(i = 1, 2, \ldots\ldots \kappa)$ *is the neighborhood of a singular point of order* $\leqq m$;

2) *each of the neighborhoods of* 1) *having been determined arbitrarily small, the region* $\tau_j\,(i = \kappa + 1, \ldots\ldots \nu)$ *is then a regular piece of an analytic surface, represented in its whole extent by a single set of parametric formulae of the type* $(A)$.

*By the neighborhood of the curve*

$$x + p(y) = 0, \qquad z = 0,$$

*is meant the set of points* $(x, y, z)$ *satisfying the relation*

$$|y| < h, \qquad |z| < \delta, \qquad |x + p(y)| < \epsilon.$$

## B. — Proof of the Lemma.

2. To prove the lemma we begin by expressing equation $(\alpha)$ by means of $(\beta)$ in the form

$$\Phi(x, y, z) = [x + p(y)]^m E(x, y) + z\Psi(x, y, z) = 0, \qquad (\gamma)$$

and then making the transformation

$$x + p(y) = x_1, \qquad (\delta)$$

thus obtaining the equation

$$\Phi(x, y, z) = \Phi_1(x_1, y, z) = x_1^m E(x_1, y) + z\psi_1(x_1, y, z) = 0. \qquad (\gamma')$$

Here, the function $E(x_1, y)$ is analytic and different from zero in the neighborhood of any point $x_1 = 0$, $y = y_0$, $(|y_0| < h)$, which corresponds to the neighborhood of the point $x_0 = p(y_0)$, $y_0$, and hence $E(x_1, y)$ is analytic throughout a region including in its interior the region

$$|x_1| < \epsilon, \qquad |y| < h,$$

if the positive quantity $\epsilon$ is suitably chosen. A similar remark applies to the analytic character of the function $\psi_1(x_1, y, z)$, and hence $\Phi_1(x_1, y, z)$ is an analytic function of its three arguments throughout a region including in its interior the region

$$|x_1| < \epsilon, \qquad |y| < h, \qquad |z| < \delta.$$

Now express equation $(\gamma')$ in the form

$$\Phi_1(x_1, y, z) = \Sigma p_{rs}(y) x_1^r z^s + F(x_1, y, z) = 0, \qquad (\epsilon)$$

where

$$0 < r + s = m_1 \leqq m,$$

$m_1$ being the lowest degree of any term in $x_1$ and $z$ together, and $F(x_1, y, z)$ including all terms of degree higher than $m_1$ in the two variables $x_1$, $z$. Each coefficient $p_r(y)$ may be divisible by a power of $y$, $y^l$. In that case, however, $m_1$ must be less than $m$, for the term in ${x_1}^m$ is present in $\Phi_1(x_1, y, z)$.

By means of a transformation with non-vanishing determinant,

$$\left.\begin{aligned} x_1 &= a_1 x_2 + \beta_1 z_2 \\ z &= a_2 x_2 + \beta_2 z_2 \end{aligned}\right\}, \qquad (\zeta)$$

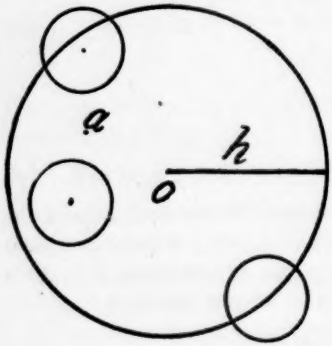
$\Phi_1$ can be thrown into the form:

$$\Phi_1(x_1, y, z) = \Phi_2(x_2, y, z_2) =$$

$$q_0(y){x_2}^{m_1} + q_1(y){x_2}^{m_1-1}z_2 + \ldots\ldots + q_{m_1}(y){z_2}^{m_1} + F_2(x_2, y, z_2) = 0 \quad (\eta)$$

where $q_0(y) \not\equiv 0$.

Consider first the points of the circle $|y| \leqq h$ at which $q_0(y) = 0$, if such exist. Each one of these points $y_i$ $(i = 1, 2, \ldots\ldots \kappa)$ is a singular point of $\Phi_2 = 0$ of order not greater than $m$, and its neighborhood




$$|x_1| < \epsilon, \quad |y - y_i| < \epsilon, \quad |z| < \delta$$

may be chosen arbitrarily small. Surround each of these points in the circle $|y| = h$ by a circle of arbitrarily small radius $\epsilon'$. We now proceed to consider the region about an arbitrary point $a$ of the circle $|y| \leqq h$ not lying in any of the regions just cut out. Let

$$y_2 = y - a$$

and let $\Phi_2$ then be written in the form

$$\Phi_2(x_2, y, z_2) = \bar{\Phi}_2(x_2, y_2, z_2) =$$

$$\bar{q}_0(y_2){x_2}^{m_1} + \bar{q}_1(y_2){x_2}^{m_1-1}z_2 + \ldots\ldots + \bar{q}_{m_1}(y_2){z_2}^{m_1} + \bar{F}_2(x_2, y_2, z_2)$$
$$= [{x_2}^{m_1} + r_1(y_2){x_2}^{m_1-1}z_2 + \ldots\ldots + r_{m_1}(y_2){z_2}^{m_1}]\,E(y_2) + \bar{F}_2(x_2, y_2, z_2)$$
$$= 0. \qquad (\theta)$$

3. Apply to the function $\bar{\Phi}_2$ the quadratic transformation

$$x_2 = x_3 z_2.^*$$

---

* Here, for the first time, a quadratic transformation of the type that transforms but a single variable is employed. Such transformations do not occur in Kobb's analysis. They appear to be indispensable.

Let the result be written as follows: —

$$\bar{\Phi}_2(x_2, y_2, z_2) = z_2^{m_1}\Phi_2(x_2, y_2, z_2) =$$

$$z_2^{m_1}\{[x_3^{m_1} + r_1(y_2)x_3^{m_1-1} + \ldots\ldots + r_{m_1}(y_2)]E(y_2) + z_2F_2(x_3, y_2, z_2)\} = 0. \quad (\kappa)$$

From this last equation we deduce the following theorem: —

*All points of the surface* $\Phi = 0$ *in the neighborhood of the curve*

$$\phi(x, y) = 0, \qquad z = 0,$$

*are mapped upon a finite number of new neighborhoods which are*

1) *neighborhoods of singular points of degree* $\leqq m$, *which neighborhoods may be taken arbitrarily small;*

2) *neighborhoods of new multiple curves on surfaces constituted like the surface* $\Phi(x, y, z) = 0$ *of the lemma, the values of* $m$ *thus arising never exceeding the original* $m$ *of the lemma.*

By the same kind of reasoning as in § 1, 5, we show namely that for any one of the above values of $a$, the corresponding value of $y_2$ being in or on the circle of convergence of the Taylor's development about the point $z_2 = 0$ of the function

$$r_\lambda(y_2), \qquad \lambda = 1, 2, \ldots\ldots m_1,$$

all points of the surface $\Phi_2 = 0$ in the neighborhood of the curve

$$\phi(x, y) = 0, \qquad z = 0,$$

are represented by points in the neighborhoods of points of the curve

$$x_3^{m_1} + r_1(y_2)x_3^{m_1-1} + \ldots\ldots + r_{m_1}(y_2) = 0, \qquad z_2 = 0,$$

on the surface $\Phi_3 = 0$, i. e. if such a value of $y$ is $b$, so that the corresponding value of $y_2$ is $(b - a)$, and if the roots of the equation

$$x_3^{m_1} + r_1(b - a)x_3^{m_1-1} + \ldots\ldots + r_{m_1}(b - a) = 0 \qquad (\mu)$$

are $a_1, a_2, \ldots\ldots a_{m_1}$ then points of the surface $\Phi_2 = 0$ for which

$$|x_2| < \delta, \qquad |z_2| < \delta, \qquad y = b,$$

are connected with the points of the surface $(\kappa)$ by the relation

$$x_2 = z_2(x_\sigma + a_\sigma), \qquad y_2 = b - a, \qquad \sigma = 1, 2, \ldots\ldots m_1.$$

Further, if we limit $y_2$ to a circle not reaching out to the nearest point for which $\bar{q}_0(y_2)$ vanishes, we have an upper limit for $a_\sigma$ as a root of the equation $(\mu)$, and thus by taking $z_2$ and $x_\sigma$ small enough we can make $x_2$ as small as we please. Then the transformations $(\delta)$ and $(\zeta)$ still

secure a limit for the values of $x$ and $z$, and thus we have represented a neighborhood of the curve

$$\phi(x, y) = 0, \qquad z = 0,$$

on the surface

$$\Phi(x, y, z) = 0$$

as required.

Now, however small the neighborhood we shut off about the points in the region $|y| < h$ for which $q_0(y)$ vanishes, since the results established above would hold also in a circle of radius $h_1 > h$, but still less than the radius of convergence of the series for $p(y)$ in $(\beta)$, we can fill up the remainder of the circle of radius $h$ with circles within which $q_0(y)$ does not vanish, these circles overlapping at all points the boundaries of the excepted neighborhoods and not reaching up to the excepted points. Within each of these circles we have a development of type $(\kappa)$. Consider one of these new circles. We want to consider the neighborhood of the curve

$$\phi_3(x_3, y_2) = x_3^{m_1} + r_1(y_2)\, x_3^{m_1-1} + \ldots\ldots + r_{m_1}(y_2) = 0. \qquad (\nu)$$

If this is a multiple curve of the $m_1$-th order and $m_1 < m$, we have reduction. Moreover, if $m_1 = m$, but

$$x_3^{m_1} + r_1(y_2)\, x_3^{m_1-1} + \ldots\ldots + r_{m_1}(y_2) \neq [x_3 + p_3(y_2)]^{m_1},$$

we also have reduction. We need consider, therefore, only the case that

$$\left.\begin{aligned} x_3^{m_1} + r_1(y_2)\, x_3^{m_1-1} + \ldots\ldots + r_{m_1}(y_2) &= [x_3 + p_3(y_2)]^{m_1}, \\ m_1 &= m, \end{aligned}\right\} \qquad (\nu')$$

and show that this case can repeat itself at most but a finite number of times.

4. Suppose the function $\phi_3(x_3, y_2)$ has the form $(\nu')$. Apply to the surface $\Phi_3(x_3, y_2, z_2) = 0$, $(\kappa)$, the transformation

$$x_3 + p_3(y_2) = x_4,$$

and reduce the result to the form

$$\Phi_4(x_4, y_2, z_2) = x_4^{m_1} E(y_2) + z_2 F_4(x_4, y_2, z_2) = (0). \qquad (o)$$

If any term in $z_2 F_4(x_4, y_2, z_2)$ is of degree in $x_4$ and $z_2$ together less than $m_1$, it appears at once that we have a line of lower order. So we assume there are no such terms. Also, as the coefficient of $x_4^{m_1}$ does not vanish identically in $y_2$ (in fact, not at all) no transformation of

type ($\zeta$) is needed, and after collecting all terms of the $m_1$-th order we make at once the substitution

$$x_4 = x_5 x_2,$$

and proceed in the same manner as before if the degree is not reduced. For convenience, we suppose the succession of multiple curves of the same order to begin with that on the surface $\Phi = 0$, and use a notation independent of that hitherto employed.

Our successive transformations are of the type

$$\left.\begin{array}{lll} x & -p_1(y) = \bar{x}_1 & \bar{x}_1 = x_1 z \\ x_1 & -p_2(y) = \bar{x}_2 & \bar{x}_2 = x_2 z \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ x_{\nu-1} & -p_\nu(y) = \bar{x}_\nu & \bar{x}_\nu = x_\nu z \end{array}\right\} \qquad (\pi)$$

whence

$$x = p_1(y) + p_2(y) z + p_3(y) z^2 + \ldots\ldots + p_\nu(y) z^{\nu-1} + x_\nu z^\nu. \qquad (\rho)$$

Develop the function $\Phi$ in ($\gamma$) by Weierstrass's Theorem:

$$\begin{aligned} \Phi(x, y, z) &= [x^m + q_1(y, z) x^{m-1} + \ldots\ldots + q_m(y, z)] E(x, y, z) \\ &= F(x, y, z) E(x, y, z). \end{aligned}$$

From ($\rho$) we derive the relation:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x_\nu}\frac{\partial x_\nu}{\partial x} = \frac{1}{z^\nu}\frac{\partial F}{\partial x_\nu}. \qquad (\sigma)$$

The succession of transformations ($\pi$) so long as it does not reduce the degree in $x$ and $z$, takes out of the $F$ factor at each step the factor $z^\nu$, since, on account of the constant term in the $E$ factor, no power of $z$ could come out of it. So, after the $\nu$ transformations ($\pi$), we have

$$\begin{aligned} F(x, y, z) &= z^{\nu m} F_\nu(x_\nu, y, z) = \\ z^{\nu m}&[x_\nu^{\,m} + q_{1\nu}(y, z) x_\nu^{\,m-1} + \ldots\ldots + q_{m\nu}(y, z)], \end{aligned} \qquad (\tau)$$

and by ($\sigma$)

$$\frac{\partial F}{\partial x} = z^{\nu(m-1)}\frac{\partial F_\nu}{\partial x_\nu}.$$

Now we may consider $F$ as having no multiple factors vanishing at the point (0, 0, 0). So we have the relation

$$L(x, y, z) F + M(x, y, z)\frac{\partial F}{\partial x} = R(y, z) = z^\lambda R_1(y, z) \qquad (\upsilon)$$

where

$$R(y, z) \not\equiv 0, \qquad R_1(y, 0) \not\equiv 0.$$

Substitute in equation $(\nu)$ for $x$ from $(\rho)$, using for $F$ and $\frac{\partial F}{\partial x}$ their values as derived above, and we have

$$z^{\nu m} L_\nu(x_\nu, y, z) F_\nu + z^{\nu(m-1)} M_\nu(x_\nu, y, z) \frac{\partial F_\nu}{\partial x_\nu} = z^\lambda R_1(y, z).$$

The left side of this equation is divisible by $z^{\nu(m-1)}$ and so the right side must be also.

$$\therefore \qquad \nu(m-1) \leqq \lambda,$$

and we have an upper limit for $\nu$, the number of transformations which leave the order of the multiple line unchanged.

The securing of the regions of class 2) in 1, follows from the reduction just proved. If for all multiple curves of order $n$ or less the lemma is assumed to hold, this reduction establishes it for all curves of order $n + 1$, since by it the neighborhoods are represented by those of lower order. But we know it to be true for curves of the first order, and so by mathematical induction we establish it for curves of all orders.

5. *The neighborhoods of singular points in* 3, *if they are of the m-th order can be taken along the curve*

$$\phi(x, y) = 0, \qquad z = 0,$$

*on the surface*

$$\Phi(x, y, z) = 0.$$

In fact, the first lot of points excepted, those for which in equation $(\eta)$ $q_0(y)$ vanishes, are along the line

$$x_2 = 0, \qquad z_2 = 0,$$

which is connected with the original curve by the one-to-one transformations $(\delta)$ and $(\zeta)$. Also so long as the multiple curve does not break up into simpler curves, the neighborhoods correspond, and when this reduction takes place we can cut out the neighborhoods of the points common to all of the resulting curves by cutting out neighborhoods along the original curve for the same values of $y$.

## C. — The Reduction of the Original Singularity.

The transformations hitherto considered, when applied to the original surface $\Phi(\xi, \eta, \zeta) = 0$, make it possible to map the neighborhood of the point (0, 0, 0) of that surface on a finite number of regions which are of two classes: —

1) neighborhoods of singular points of transformed surfaces;

2) regular pieces of transformed surfaces.

The pieces of class 2) lead at once to representation by means of parametric formulae of type (A). The singular points of class 1) are all of lower order than the original singularity except in one case, and it is this case that it remains to consider in §§ 3, 4. The case can present itself at the outset only if the polynomial $(\xi, \eta, \zeta)_m$ is the product of $m$ linear factors in $\xi, \eta, \zeta$, all vanishing for a single set of values of the arguments $\xi, \eta, \zeta$ not all zero. Geometrically, the tangent cone, $(\xi, \eta, \zeta)_m = 0$, of the surface $\Phi(\xi, \eta, \zeta) = 0$ at the point (0, 0, 0) consists of $m$ planes having a common line of intersection. It is found necessary to distinguish two sub-cases according to whether the planes themselves are not all coincident, or are all coincident.

To sum up, then, we already have reduction in all cases except when we are led to singular points in class 1) of the particular type just described.

## § 3.

### A. — The Singular Points of Special Type.

1. In the special case in which the function $(\xi, \eta, \zeta)_m$ is composed of $m$ linear factors, each vanishing for all points on a common line, it is possible to reduce the singularity by means of a finite succession of quadratic transformations together with certain additional transformations.

We consider two cases: —

Case A. — The $m$ linear factors of $(\xi, \eta, \zeta)_m$ are not all equal.

Case B. — The $m$ linear factors of $(\xi, \eta, \zeta)_m$ are all equal.

2. Case A. — $(\xi, \eta, \zeta)_m$ is composed of $m$ linear factors not all equal.

The surface can be expressed in the form

$$\Phi(\xi, \eta, \zeta) = (\xi, \eta)_m + (\xi, \eta, \zeta)_{m+1} + \ldots\ldots = 0 \qquad (13)$$

where $(\xi, \eta)_m$ contains terms in both $\xi^m$ and $\eta^m$.

If the surface were in a form

$$f(u, v, w) = (u, v, w)_m + (u, v, w)_{m+1} + \ldots\ldots = 0$$

with the condition that the $m$ linear factors of $(u, v, w)_m$ all vanish for the line

$$u = \alpha w, \qquad v = \beta w,$$

we could make the transformation

$$\xi = u - \alpha w, \qquad \eta = v - \beta w, \qquad \zeta = w,$$

and all the resulting linear factors would have to vanish when

$$\xi = 0, \qquad \eta = 0,$$

and so not contain $\zeta$.

Also by a linear homogeneous transformation in $\xi$ and $\eta$ we can secure the presence of terms in $\xi^m$ and $\eta^m$, and in such case every linear factor of $\phi(\xi, \eta)$, which here is $(\xi, \eta)_m$ itself, will contain $\xi$ and thus secure condition 3) of § 1, 3.

## B. — Quadratic Transformations.

3. The succession of surfaces and corresponding quadratic transformations which are applied to the new singular points as found, so long as they do not reduce the degree, can be written in the form

$$\left.\begin{array}{lll}
\Phi(\xi, \eta, \zeta) = \zeta^m \Phi_1(\xi_1, \eta_1, \zeta), & \xi = \zeta(\xi_1 + \gamma_1), & \eta = \zeta(\eta_1 + \delta_1), \\
\Phi_1(\xi_1, \eta_1, \zeta) = \zeta^m \Phi_2(\xi_2, \eta_2, \zeta), & \xi_1 = \zeta(\xi_2 + \gamma_2), & \eta_1 = \zeta(\eta_2 + \delta_2), \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\Phi_{\nu-1}(\xi_{\nu-1}, \eta_{\nu-1}, \zeta) = \zeta^m \Phi_\nu(\xi_\nu, \eta_\nu, \zeta), & \xi_{\nu-1} = \zeta(\xi_\nu + \gamma_\nu), & \eta_{\nu-1} = \zeta(\eta_\nu + \delta_\nu).
\end{array}\right\} \quad (14)$$

Apply to the surface (13) the transformation

$$\xi = \xi_1 \zeta, \qquad \eta = \eta_1 \zeta,$$

and we have

$$\begin{aligned}
\Phi(\xi, \eta, \zeta) &= \zeta^m [(\xi_1, \eta_1)_m + \zeta(\xi_1, \eta_1, 1)_{m+1} + \cdots\cdots] \\
&= \zeta^m [(\xi_1, \eta_1)_m + \zeta\psi(\xi_1, \eta_1, \zeta)] \qquad (15) \\
&= \zeta^m \Phi_1(\xi_1, \eta_1, \zeta).
\end{aligned}$$

As we assume the transformation does not reduce the degree of the singular point, there can be no term of degree less than $m$ in the part $\zeta\psi(\xi_1, \eta_1, \zeta)$ and as all terms of this contain $\zeta$, when we put the expression in the form

$$\Phi_1(\xi_1, \eta_1, \zeta) = (\xi_1, \eta_1, \zeta)_m + (\xi_1, \eta_1, \zeta)_{m+1} + \cdots\cdots \qquad (16)$$

we will secure reduction by another quadratic transformation unless $(\xi_1, \eta_1, \zeta)_m$ is the product of $m$ linear factors with a common line of intersection. In this case the factors cannot be all equal, for then $(\xi_1, \eta_1, 0)_m$ would have its linear factors all equal, but these are the factors of $(\xi_1, \eta_1)_m$. Also the common point of intersection of the lines

in which the plane $\zeta = 1$ cuts the planes corresponding to these factors is at a finite distance. We have now the conditions 2) and 3) of § 1, 3, and are ready to apply the transformations

$$\xi_1 = \bar{\xi}_1 \zeta, \qquad \eta_1 = \bar{\eta}_1 \zeta,$$

giving

$$\Phi_1(\xi_1, \eta_1, \zeta) = \zeta^m [(\bar{\xi}_1, \bar{\eta}_1, 1)_m + \zeta(\bar{\xi}_1, \bar{\eta}_1, 1)_{m+1} + \ldots\ldots] = 0. \quad (17)$$

Now if $\bar{\xi}_1 = \gamma_2, \bar{\eta}_1 = \delta_2$, is the common point for which the $m$ factors of $(\bar{\xi}_1, \bar{\eta}_1, 1)_m$ vanish, then the substitution

$$\xi_2 = \bar{\xi}_1 - \gamma_2, \qquad \eta_2 = \bar{\eta}_1 - \delta_2,$$

gives a group of $m$th degree terms in $\xi_2$ and $\eta_2$ exactly corresponding to the terms of $(\xi, \eta)_m$. So in the successive collection of terms of the $m$th degree, the terms of $(\xi, \eta)_m$ are always carried over with merely a change of subscript, and thus we never introduce the condition of $m$ equal linear factors. Accordingly so long as the degree of the singular point is not reduced, the intermediate transformations are of the type

$$\xi_\mu = \bar{\xi}_\mu \zeta, \qquad \eta_\mu = \bar{\eta}_\mu \zeta,$$
$$\xi_{\mu+1} = \bar{\xi}_\mu - \gamma_{\mu+1}, \qquad \eta_{\mu+1} = \bar{\eta}_\mu - \delta_{\mu+1},$$

thus securing the succession of transformations (14).

4. The succession of transformations in 3 will lead to the relation

$$\xi^{m\nu} L(\xi_\nu, \eta_\nu, \zeta) \bar{\Phi}_\nu + M(\xi_\nu, \eta_\nu, \zeta) \zeta^{(m-1)\nu} \frac{\partial \bar{\Phi}_\nu}{\partial \xi_\nu} = R(\eta, \xi) \not\equiv 0 \quad (18)$$

where

$$\Phi_\nu(\xi_\nu, \eta_\nu, \zeta) = \bar{\Phi}_\nu(\xi_\nu, \eta_\nu, \zeta) E(\xi_\nu, \eta_\nu, \zeta).$$

Combining transformations (14) we have the relations

$$\left.\begin{aligned} \xi &= \gamma_1 \zeta + \gamma_2 \zeta^2 + \ldots\ldots + \gamma_\nu \zeta^\nu + \zeta^\nu \xi_\nu \\ \eta &= \delta_1 \zeta + \delta_2 \zeta^2 + \ldots\ldots + \delta_\nu \zeta^\nu + \zeta^\nu \eta_\nu \end{aligned}\right\} \quad (19)$$
$$\Phi(\xi, \eta, \zeta) = \zeta^{m\nu} \Phi_\nu(\xi_\nu, \eta_\nu, \zeta)$$

As $\Phi$ contains both $\xi^m$ and $\eta^m$ terms, we can develop by Weierstrass's Theorem.

$$\left.\begin{aligned} \Phi(\xi, \eta, \zeta) &= [\xi^m + p_1(\eta, \zeta)\xi^{m-1} + \ldots\ldots + p_m(\eta, \zeta)] E_1(\xi, \eta, \zeta) \\ &= \bar{\Phi}(\xi, \eta, \zeta) E_1(\xi, \eta, \zeta) \\ \Phi(\xi, \eta, \zeta) &= [\eta^m + q_1(\xi, \zeta)\eta^{m-1} + \ldots\ldots + q_m(\xi, \zeta)] E_2(\xi, \eta, \zeta) \\ &= \bar{\Psi}(\xi, \eta, \zeta) E_2(\xi, \eta, \zeta). \end{aligned}\right\} \quad (20)$$

As the function $E_1(\xi, \eta, \zeta)$ contains a constant term, when the first transformation of (14) is made, the factor $\zeta^m$ must come out of the $\bar{\Phi}$, and a

similar result is true for all of the succeeding transformations. So in the first part of (14) we could write $\bar{\Phi}$ for $\Phi$, $\bar{\Phi}_\mu$ for $\Phi_\mu$, $(\mu = 1, 2, \ldots\ldots \nu)$, where the $\bar{\Phi}$'s are derived successively in the same way as the $\Phi$'s. At each stage the $\bar{\Phi}$ factor must contain all the terms of lowest degree in the corresponding $\Phi$ (except for a constant multiple), and no lower terms; for, otherwise, either there would be lower terms in the product by the corresponding $E$ factor on account of its constant term, or the required terms would not be present.

Now, by (19),

$$\frac{\partial \bar{\Phi}}{\partial \xi} = \frac{\partial \bar{\Phi}}{\partial \xi_\nu}\frac{\partial \xi_\nu}{\partial \xi} = \frac{1}{\zeta}\frac{\partial \bar{\Phi}}{\partial \xi_\nu}. \tag{21}$$

Also

$$\bar{\Phi} = \zeta^{m\nu}\bar{\Phi}_\nu; \tag{22}$$

and, combining with (21), we have

$$\frac{\partial \bar{\Phi}}{\partial \xi} = \zeta^{(m-1)\nu}\frac{\partial \bar{\Phi}_\nu}{\partial \xi_\nu}. \tag{23}$$

But as $\Phi$ has no multiple factors vanishing at (0, 0, 0) (see § 1, 3), we have the relation

$$L(\xi, \eta, \zeta)\bar{\Phi} + M(\xi, \eta, \zeta)\frac{\partial \bar{\Phi}}{\partial \xi} = R(\eta, \zeta) \not\equiv 0. \tag{24}$$

Then, substituting for $\xi$ and $\eta$ from (19) on the left side of equation (24) and using the relations (22) and (23), we have the required relation (18).

5. If $\nu$ is taken large enough the transformations (14) will lead to the relations

$$L_\nu(\xi_\nu, \eta_\nu, \zeta)\bar{\Phi}_\nu + M_\nu(\xi_\nu, \eta_\nu, \zeta)\frac{\partial \bar{\Phi}_\nu}{\partial \xi_\nu} = \zeta^{r_1}[\eta_\nu + \omega_1(\zeta)]^{r_1}E(\eta_\nu, \zeta), \tag{25}$$

$$P_\nu(\xi_\nu, \eta_\nu, \zeta)\Psi_\nu + Q_\nu(\xi_\nu, \eta_\nu, \zeta)\frac{\partial \bar{\Psi}}{\partial \eta_\nu} = \zeta^{r_2}[\xi_\nu + \omega_2(\zeta)]^{r_2}E(\xi_\nu, \zeta), \tag{26}$$

where

$$\Phi_\nu(\xi_\nu, \eta_\nu, \zeta) = \bar{\Phi}_\nu(\xi_\nu, \eta_\nu, \zeta)E_1(\xi_\nu, \eta_\nu, \zeta) = \bar{\Psi}_\nu(\xi_\nu, \eta_\nu, \zeta)E_2(\xi_\nu, \eta_\nu, \zeta). \tag{27}$$

We consider the effect of the transformations (14) on $R$ in (18). Express it in the form

$$R(\eta, \zeta) = \zeta^r[(\eta, \zeta)_n + (\eta, \zeta)_{n+1} + \ldots\ldots] = \zeta^r S(\eta, \zeta),$$

where

$$S(\eta, 0) \not\equiv 0.$$

If $(\eta, \zeta)_n$ contains no term in $\eta$, the first transformation of (14) will allow the factor $\zeta^n$ to be taken out of $S$, leaving behind a constant term, and thus securing the form (25) at once with

$$q_1 = p + n, \qquad r_1 = 0.$$

Next, suppose $(\eta, \zeta)_n$ does contain terms in $\eta$, but no term in $\eta^n$, i. e. we can express it in the form

$$(\eta, \zeta)_n = \eta^r \zeta^s (\eta, \zeta)_{n-r-s},$$

where $(\eta, \zeta)_{n-r-s}$ contains terms in both $\eta^{n-r-s}$ and $\zeta^{n-r-s}$ and $s > 0$. Then if any transformation

$$\eta = \zeta(\bar{\eta} + \delta)$$

is applied, there can be divided out of $(\eta, \zeta)_n$ the factor $\zeta^n$ leaving behind as the term of highest degree one in $\bar{\eta}^{n-s}$. This cannot be cancelled with any term from another part $(\eta, \zeta)_{n+b}$ for any term from this would have as a factor $\zeta^l$ after the $\zeta^n$ has been divided out. As long, then, as the $\eta$ variable does not enter to the highest degree in the expression corresponding to $(\eta, \zeta)_n$ if $n > 0$, the degree of the $S$ factor is decreased with each transformation, while the exponent of $\zeta$ outside may be increased. Accordingly, by a finite number of transformations, we reduce the $S$ factor either to an $E$ function or to an expression in which the $\eta$ variable enters to the highest degree in the collection of terms of lowest order. In the former case we have the form required. In the latter case, suppose for convenience that this condition holds for the function $S(\eta, \zeta)$. By Weierstrass's Theorem we develop in the form

$$\begin{aligned} S(\eta, \zeta) &= [\eta^n + r_1(\zeta)\eta^{n-1} + \cdots\cdots + r_n(\zeta)] E(\eta, \zeta) \\ &= T(\eta, \zeta) E(\eta, \zeta). \end{aligned} \tag{28}$$

Consider the $n$ factors of $T(\eta, \zeta)$,

$$T(\eta, \zeta) = \prod_{\lambda=1}^{n} [\eta + s_\lambda(\zeta)]. \tag{29}$$

If the factors are not all equal, pair them off, so that in each pair there will be two different factors, leaving a number of equal factors:

$$\{[\eta + s_{t1}(\zeta)][\eta + s_{u1}(\zeta)]\} \cdots\cdots \{[\eta + s_{t_k}(\zeta)][\eta + s_{u_k}(\zeta)]\} [\eta + s_\nu(\zeta)]^l. \tag{30}$$

Now, for each pair,

$$N_k = [\eta + s_{t_k}(\zeta)][\eta + s_{u_k}(\zeta)],$$

we have the relation

$$N_k + P_k(\eta, \zeta) \frac{\partial N_k}{\partial \eta} = L_k(\zeta) \not\equiv 0, \tag{31}$$

since the two factors are unequal. Then, by the same reasoning as used

in 4, the succession of transformations (14) which leaves the degree of $T$ unchanged will secure for equation (31) a form

$$\zeta^{2\nu} N_{k\nu}(\eta_\nu, \zeta) + \zeta^{\nu} P_{k\nu}(\eta_\nu, \zeta) \frac{\partial N_{k\nu}}{\partial \eta_\nu} = L_k(\zeta) = \zeta^{\lambda} E(\zeta).$$

The left side of the equation is divisible by $\zeta^{\nu}$, and so the right side must be,

$$\therefore \qquad \nu \leqq \lambda,$$

and we have an upper limit for $\nu$, the number of transformations which leave the factor $N_k$ of the second degree, and as a result leave the function $T$ of the $n$th degree. So, unless the function $T(\eta, \zeta)$ in (28) is composed of $n$ equal factors of form

$$[\eta + s(\zeta)]^n, \tag{32}$$

the transformation of (14) will finally reduce its degree. Then, by applying the same reasoning to the resulting function, we see that finally the function corresponding to $S(\eta, \zeta)$ either becomes an $E$ function or has besides the $E$ factor a factor of form (32), thus securing the form (25) if we divide out the factor $\zeta^{(m-1)\nu}$.

The condition (26) is secured by using on the second equation in (20) the same kind of reasoning as applied in 4 and 5. Then we take for $\nu$ the larger of the two values required to secure conditions (25) and (26).

### C. — Further Transformations.

6. A transformation

$$\left.\begin{aligned} \xi_\nu &= \bar{\xi}_\nu - \omega_2(\zeta) \\ \eta_\nu &= \bar{\eta}_\nu - \omega_1(\zeta) \end{aligned}\right\} \tag{33}$$

applied to the surface

$$\Phi_\nu(\xi_\nu, \mu_\nu, \zeta) = 0$$

in 5 will secure a form in which the singularity will be reduced by either

1) a further succession of transformations as in 3,

2) the method of the Lemma, § 2.

Let us consider here the case in which either $r_1$ or $r_2$ in (25) and (26) is zero. Then in one of the equations a further succession of transformations of type (14) will not change the power of $\zeta$ as a factor on the right; and if there are $\mu$ such further transformations, the reasoning

of 4 shows that the left side becomes divisible by $\zeta^{(m-1)\mu}$. So we have either

$$(m-1)\mu \leqq q_1 \qquad \text{or} \qquad (m-1)\mu \leqq q_2$$

and thus an upper limit for $\mu$, the number of transformations which leave the order of the singular point unchanged.

Now, to consider the transformation (33), we see that it is a one-to-one transformation by which the surface remains analytic near the origin. $\omega_1(\zeta)$ and $\omega_2(\zeta)$ contain no constant term, for otherwise the factor

$$\eta_\nu + \omega_1(\zeta) \qquad \text{or} \qquad \xi_\nu + \omega_2(\zeta)$$

could be combined with the $E$ factor. Then the transformation (33) leaves the $E$ factors still $E$ factors, and the factors vanishing at the origin still vanishing there. Also, it is easily seen that this transformation leaves the terms of type $(\zeta, \eta)_m$ still in the part $(\xi_\nu, \eta_\nu, \zeta)_m$. Further, if the function $\Phi_\nu(\xi_\nu, \eta_\nu, \zeta)$ goes over into $X(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)$, we have

$$\frac{\partial \Phi_\nu}{\partial \xi_\nu} = \frac{\partial X}{\partial \xi_\nu} = \frac{\partial X}{\partial \bar{\xi}_\nu}\frac{\partial \bar{\xi}_\nu}{\partial \xi_\nu} = \frac{\partial X}{\partial \xi_\nu}$$

and similar conditions hold for the partial derivative with reference to $\eta_\nu$. Accordingly, if by the transformation (33) $\Phi_\nu(\xi_\nu, \eta_\nu, \zeta)$ goes over into $\Omega(\xi_\nu, \eta_\nu, \zeta)$ we replace equations (25), (26), and (27) by

$$L_\nu(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)\, X(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta) + \bar{M}_\nu(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)\frac{\partial X}{\partial \bar{\xi}_\nu} = \zeta^{q_1}\bar{\eta}_\nu^{r_1} E(\bar{\eta}_\nu, \zeta), \quad (34)$$

$$P_\nu(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)\, H(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta) + \bar{Q}_\nu(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)\frac{\partial H}{\partial \bar{\eta}_\nu} = \zeta^{q_2}\bar{\xi}^{r_2} E(\bar{\xi}_\nu, \zeta), \quad (35)$$

$$\Omega(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta) = X(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)\, \bar{E}_1(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta) = H(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)\, \bar{E}_2(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta). \quad (36)$$

Now, in a further succession of transformations of type (14) on the surface $\Omega(\xi_\nu, \eta_\nu, \zeta) = 0$, if there enters either a $\gamma$ or a $\delta$ not 0, then on the right side of equation (34) or (35) the only factor remaining outside of the $E$ factor is a power of $\zeta$, and we must finally have a reduction as shown above. So it is only in the case in which all the $\gamma$'s and $\delta$'s of the later transformations are 0 that we are not already sure of reducing the singularity. Now if in $\Omega(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)$ there is any term of degree less than $m$ in $\bar{\xi}_\nu$ and $\bar{\eta}_\nu$ combined, such a succession of transformations must reduce this term to a degree less than $m$ and thus reduce the singularity.

For suppose such a term to be $a\bar{\xi}_\nu^f \bar{\eta}_\nu^g \zeta^h$, where $f+g<m$. Then, by a succession of $\rho$ transformations such as defined, we have

$$\bar{\xi}_\nu = \zeta^\rho \xi_{\nu+\rho}, \qquad \bar{\eta}_\nu = \zeta^\rho \eta_{\nu+\rho},$$

(derived from form of (19) when all $\gamma$'s and $\delta$'s are 0). Substituting this in the expression above we get

$$a\xi_{\nu+\rho}^f \eta_{\nu+\rho}^g \zeta^{h+\rho(f+g)}$$

But we must divide out of this $\zeta^{m\rho}$, so that we have left the term

$$a\xi_{\nu+\rho}^f \eta_{\nu+\rho}^g \zeta^{h+\rho(f+g-m)}.$$

This term could not combine with any other derived in a similar way, for if we had another term $b\bar{\xi}_\nu^f \bar{\eta}_\nu^g \zeta^k$, we should get

$$b\xi_{\nu+\rho}^f \eta_{\nu+\rho}^g \zeta^{k+\rho(f+g-m)}$$

and this would not combine with the other unless $k=h$. Now, if the degree of the singular point is not reduced, we must have for the sum of the exponents

$$f+g+h+\rho(f+g-m) \geqq m$$

$$\text{or } (\rho+1)(m-f-g) \leqq h,$$

$$\text{and as } m>f+g$$

$$\rho+1 \leqq \frac{h}{m-f-g}$$

thus securing an upper limit for $\rho$, the number of transformations which leave the term and the singular point of the $m$th order.

So it is only in the case in which all terms of $\Omega(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)$ are of degree not less than $m$ in $\bar{\xi}^\nu$ and $\bar{\eta}_\nu$ together that we do not have a reduction of singularity by the succession of transformations of type (14). But, in this exceptional case, we have the conditions of the Lemma of § 2, where in equation ($\theta$) we take

$$\bar{\xi}_\nu = x_2, \qquad \bar{\eta}_\nu = z_2, \qquad \zeta = y_2,$$

the singular line being

$$\bar{\eta}_\nu = 0, \qquad \bar{\xi}_\nu = 0.$$

There is in $\Omega(\bar{\xi}_\nu, \bar{\eta}_\nu, \zeta)$ a term in $\bar{\xi}_\nu^m$, and so the expression $\bar{q}_0(y_2)$ does not vanish when $y_2 = 0$. Accordingly, within a neighborhood about this point, we can break up the singularity by the methods of

§ 2. Further, since the expression $(\bar{\xi}_\nu, \bar{\eta}_\nu)_m$ is not composed of $m$ equal factors, the part

$$\bar{q}_0(0)\,x_2^{m_1} + \bar{q}_1(0)\,x_2^{m_1-1}z + \ldots\ldots + q_{m_1}(0)\,z_2^{m_1}$$

from $(\theta)$ which corresponds to $(\xi_\nu, \eta_\nu)$ is not composed of $m$ equal factors, and the resulting curve in $(\kappa)$

$$x_2^{m_1} + r_1(y_2)\,x_2^{m_1-1} + \ldots\ldots + r_{m_1}(y_2) = 0$$

has not $m$ equal roots when $y_2 = 0$. So a single transformation of the kind in § 2, 3, reduces the singularity in the neighborhood considered here.

7. The neighborhood of the original singular point is mapped upon a finite number of neighborhoods of simpler points.

At every stage the function $(\xi_\mu, \eta_\mu, \zeta)_m$ contains the terms of the type $(\xi, \eta)_m$ found in the original equation (13). So there is but one singular point of the $m$-th order in the finite region of the $\eta_\mu$-plane. Further, the equation $(\xi_\mu, 1, \zeta)_m = 0$ for the value $\zeta = 0$ cannot have $m$ equal roots since $(\xi, \eta)_m$ is not a perfect $m$-th power of a linear factor. Accordingly, the transformation corresponding to (8) in § 1, 4, cannot produce a singular point of the $m$-th order. So, at each step, the neighborhood of the singular point is represented by a number of regions as in § 2, C, in which but one of the points of class 1) is of the $m$th order. Further, the extra transformations (33) carry the neighborhood of the singular point over into that of the new point. So, by combining all the representations, as the singularity is finally reduced, we have the original neighborhood mapped upon a finite number of regions as in § 2, C, in which all points of class 1) are of order lower than $m$.

## § 4.

### A. — The Singular Points of Special Type (*continued*).

1. Case B. — The $m$ linear factors of $(\xi, \eta, \zeta)_m$ are all equal.

The surface can be expressed in the form

$$\begin{aligned}\Phi(\xi, \eta, \zeta) &= [\xi^m + p_2(\eta, \zeta)\,\xi^{m-2} + \ldots\ldots + p_m(\eta, \zeta)]\,E(\xi, \eta, \zeta)\\ &= X(\xi, \eta, \zeta)\,E(\xi, \eta, \zeta) = 0,\end{aligned} \tag{37}$$

where, in $X$, $\xi^m$ is the only term of degree $m$.

If it were in the form

$$f(u, v, w) = (\alpha u + \beta v + \gamma w)^m + (u, v, w)_{m+1} + \ldots\ldots = 0,$$

as one of the three numbers, $\alpha$, $\beta$, $\gamma$, is not zero, suppose $\alpha = 0$.

Then by a linear homogeneous transformation

$$\left.\begin{aligned} \bar{u} &= au + \beta v + \gamma w \\ \bar{v} &= \quad v \\ \bar{w} &= \quad\quad w \end{aligned}\right\}$$

we secure the form

$$\begin{aligned} f(u, v, w) &= \bar{f}(\bar{u}, \bar{v}, \bar{w}) \\ &= \bar{u}^m + (\bar{u}, \bar{v}, \bar{w})_{m+1} + \ldots\ldots \end{aligned}$$

By Weierstrass's Theorem we can express this in the form

$$f(\bar{u}, \bar{v}, \bar{w}) = [\bar{u}^m + \bar{p}_1(\bar{v}, \bar{w})\,\bar{u}^{m-1} + \ldots\ldots + \bar{p}_m(\bar{v}, \bar{w})]\,E(\bar{u}, \bar{v}, \bar{w}). \quad (38)$$

Now, in the expression

$$\bar{p}_\lambda(\bar{v}, \bar{w}), \qquad \lambda = 1, 2, \ldots\ldots m$$

there is no term of degree less than $\lambda + 1$, for otherwise on account of the constant term in the $E$ factor, there would have to be present in $\bar{f}$ a term of degree $\leqq m$ containing $\bar{v}$ or $\bar{w}$.

Make in (38) the transformation

$$\left.\begin{aligned} \bar{u} + \tfrac{1}{m}\bar{p}_1(\bar{v}\,\bar{w}) &= \xi \\ \bar{v} &= \eta \\ \bar{w} &= \zeta \end{aligned}\right\}$$

As $\bar{p}_1(\bar{v}, \bar{w})$ contains no term of degree less than 2, by the consideration above, $f$ goes over into form (37).

## B. The Quadratic Transformation.

2. The transformation

$$\xi = \bar{\xi}\zeta, \quad \eta = \bar{\eta}\zeta,$$

applied to $\Phi(\xi, \eta, \zeta)$ secures the form

$$\Phi(\xi, \eta, \zeta) = \zeta^m \bar{\Phi}(\bar{\xi}, \bar{\eta}, \zeta) = \zeta^m[\bar{\xi}^m + \zeta\bar{\Psi}(\bar{\xi}, \bar{\eta}, \zeta)]. \quad (39)$$

Here the curve $\phi(\bar{\xi}, \bar{\eta}) = 0$ becomes $\bar{\xi}^m = 0$, and so, applying the Lemma of § 2 to a circle in the $\bar{\eta}$-plane however large, we have within it but a finite number of singular points to treat further. But one such circle is needed, for by taking it large enough we can deal with all of the $\bar{\eta}$-plane outside of that circle by the transformation

$$\xi = \bar{\bar{\xi}}\eta, \qquad \zeta = \bar{\bar{\zeta}}\eta.$$

So we need to consider for further treatment only a finite number of points along the line $\bar{\xi} = 0$, and the point at infinity.

3. The quadratic transformations to be used are of two types

$$1)\quad \xi_\mu = \xi_{\mu+1}\zeta_\mu, \qquad \eta_\mu = (\eta_{\mu+1} + \delta_{\mu+1})\zeta_\mu, \tag{40}$$

$$2)\quad \xi_\mu = \xi_{\mu+1}\eta_\mu, \qquad \zeta_\mu = (\zeta_{\mu+1} + \epsilon_{\mu+1})\eta_\mu. \tag{41}$$

In a succession of transformations of type (14) we see that $\gamma_1 = 0$, since the first set of points is taken on the line $\bar{\xi} = 0$. Further, suppose after the substitution $\bar{\eta} - \delta_1 = \eta_1$ in $\bar{\Phi}$ of (39) the expression

$$(\bar{\xi}, \eta_1, \zeta)_m$$

contains terms besides the $\bar{\xi}^m$; then it cannot be composed of $m$ equal linear factors, for that would require a term containing $\bar{\xi}^{m-1}$; but no such term can arise from the factor $X$ of (37), and, on the other hand, it could not be the product of a term from $X$ by a non-constant term of the $E$ factor, for then, on account of the constant term of the $E$ factor, there would have to be present in $\bar{\Phi}$ a term of degree lower than $m$. So as soon as the function corresponding to $\phi$ of $\bar{\Phi}$ contains more than the $m$th power of the $\xi$ variable, the function corresponding to $(\xi, \eta, \zeta)_m$ is no longer the product of $m$ equal linear factors, and we have one of the cases treated earlier.

The same considerations apply to the transformations corresponding to type 2), since, when the transformation which deals with the infinite region is introduced, the first one of that order is of form

$$\xi = \bar{\bar{\xi}}\eta, \qquad \zeta = \bar{\bar{\zeta}}\eta.$$

Accordingly, the most general succession of transformations here is one in which groups of types 1) and 2) alternate. We shall call them the $\zeta$ and $\eta$ types respectively, and when a change is made from one type to the other, we shall speak of it as a *reversal of type.*

We shall treat the subject in two cases, first supposing that there is no reversal of type in the succession of transformations used, and later supposing that reversals of type occur.

## C. — Succession of Quadratic Transformations in which there is no Reversal of Type.

4. After a sufficient number of quadratic transformations the surface can be reduced to the form

$$\bar{\Phi}_\nu(\xi_\nu, \bar{\eta}_\nu, \zeta) = \bar{X}_\nu(\xi_\nu, \bar{\eta}_\nu, \zeta)\, E(\xi_\nu, \bar{\eta}_\nu, \zeta) =$$
$$[(\xi_\nu^m + \bar{\eta}_\nu^{p_\nu}\zeta^{\eta_\nu} E(\bar{\eta}_\nu, \zeta)\, \xi_\nu^{m-2} + \ldots + \bar{\eta}_\nu^{p_\nu}\zeta^{q_\nu} E(\bar{\eta}_\nu, \zeta)]\, E(\xi_\nu, \bar{\eta}_\nu, \zeta), \tag{42}$$

while all later transformations can be taken of the type

$$\xi_\mu = \xi_{\mu+1}\zeta, \qquad \eta_\mu = \eta_{\mu+1}\zeta. \tag{43}$$

After $\nu$ transformations of type (40), since there can be no interchange of terms among the coefficients of the different powers of the $\xi$ variables in the $X$ factor of (37), the surface will take the form

$$\Phi_\nu(\xi_\nu, \eta_\nu, \zeta) = X_\nu(\xi_\nu, \eta_\nu, \zeta)\, E(\xi_\nu, \eta_\nu, \zeta) =$$
$$[\xi_\nu^m + p_{2\nu}(\eta_\nu, \zeta)\,\xi_\nu^{m-2} + \cdots\cdots + p_{m\nu}(\eta_\nu, \zeta)]\, E(\xi_\nu, \eta_\nu, \zeta) = 0. \tag{44}$$

Now by the same reasoning as used for the function $R$ in § 3, 5, if $\nu$ is taken large enough, the coefficients of the powers of $\xi_\nu$ in $X_\nu$ will all be of the type

$$p_{s\nu}(\eta_\nu, \zeta) = \zeta^{q_s}[\eta_\nu + \nu_s(\zeta)]^{r_s} E(\eta_\nu, \zeta),$$
$$s = 2, 3, \ldots\ldots m.$$

For any one of the functions

$$\eta_\nu + \nu(\zeta)$$

there is a determinate succession of transformations of type

$$\eta_\mu = \zeta(\eta_{\mu+1} + \delta_{\mu+1})$$

which will leave it of the same degree after the $\zeta$ is divided out, all others reducing the degree at once, i. e., if

$$\eta_\mu + \nu(\zeta) = \eta_\mu + a_1\zeta + a_2\zeta^2 + \cdots\cdots,$$

we must take

$$\eta_\mu = \zeta(\eta_{\mu+1} - a_1),$$
$$\eta_{\mu+1} = \zeta(\eta_{\mu+2} - a_2),$$
$$\vdots$$
$$\text{etc.}$$

So, unless the factors

$$\eta_\nu + \nu_s(\zeta), \qquad s = 2, 3, \ldots\ldots m$$

are all equal, we must have finally some coefficient of a power of $\xi_\nu$ with the $\eta_\nu$ present only in the $E$ factor, and by taking $\nu$ large enough we come to a point where all the factors

$$\eta_\nu + \nu_s(\zeta), \qquad s = 2, 3, \ldots\ldots m,$$

are equal, some of them possibly having zero exponents.

Then we use the transformation

$$\eta_\nu + \nu_s(\zeta) = \bar{\eta}_\nu \tag{45}$$

and arrive at the form (42) required. Now any further transformation of type (40) in which the $\delta$ is not zero will leave the $\eta$ variable present only in the $E$ factors, so that the general term (after the first) of the function $X_\nu$ is of type

$$\zeta^{q_s} E(\bar{\eta}_\nu, \zeta)\, \xi_\nu^{m-s}, \qquad s = 2, 3, \ldots\ldots m.$$

Suppose, after this, there are $\rho$ transformations of type (40). Then the corresponding term after the factor $\zeta^{m\rho}$ has been divided out is

$$\zeta^{q_s+(m-s)\rho-m\rho} E(\eta_{\nu+\rho}, \zeta)\, \xi_{\nu+\rho}^{m-s}$$

and if this is of degree not less than $m$, as it must be if we are not to secure reduction, we have

$$m - s + q_s - \rho s \geqq m$$

or

$$\rho \leqq \frac{q_s - s}{s},$$

thus securing an upper limit for the number of transformations of type (40) which do not give reduction of singularity. Accordingly, after the form (42) is reached, it is only when all later $\delta$'s are zero that we are not sure of reduction.*

5. A sufficient number of transformations of type (43) applied to (42) secures either

1) reduction of singularity, or

2) the condition that for some term (the $r$th) of the $X$ factor

$$\left.\begin{aligned} \frac{p_r}{r} &\leqq \frac{p_s}{s} \\ \frac{q_r}{r} &\leqq \frac{q_s}{s} \end{aligned}\right\} \quad s = 2, 3, \ldots\ldots m.$$

If, for any term

$$\bar{\eta}_\nu^{p_r} \zeta^{q_r} E(\bar{\eta}_\nu, \zeta)\, \xi_\nu^{m-r}, \qquad p_r < r,$$

a transformation of type (43), after the factor $\zeta^m$ has been divided out, yields

$$\eta_{\nu+1}^{p_r} \zeta^{q_r+p_r-r} E(\eta_{\nu+1}, \zeta)\, \xi_{\nu+1}^{m-r},$$

decreasing the exponent of $\zeta$ by $r - p_r$. This decrease takes place at every such transformation, and thus the exponent of $\zeta$ must finally be

* We do not need to consider the possibility of having all the coefficients of the powers of $\xi_\nu$ lower than the $m$-th vanish, for then the function $X_\nu$ would have $m$ equal factors $\xi_\nu$ and this case has been excluded.

reduced to a value $q'_r$ less than $r - p_r$, in which case the sum of the exponents of the three variables,

$$p_r + q'_r + m - r,$$

is less than $m$ and reduction ensues. So it is only in the case in which for every term

$$p_s \geqq s, \qquad s = 2, 3, \ldots\ldots m,$$

that we are not sure of reduction. Suppose the number of transformations after this point to be $n$. Then we get for the new exponent of $\zeta$

$$q_s + n(p_s - s).$$

Now by taking $n$ large enough we can make the quotient

$$\frac{n(p_s - s) + q_s}{s}$$

have the lowest value for the term in which $\frac{p_s - s}{s}$ is lowest, while if this is the same for two or more terms, we can make the fraction above lowest for the one in which $\frac{q_s}{s}$ is lowest. Accordingly, by a finite number of transformations of type (43) we secure the condition that $\frac{p_r - r}{r}$ and so $\frac{p_r}{r}$ is lowest in the same term in which $\frac{q_r}{r}$ is lowest.

6. A succession of transformations of type

$$\xi_\mu = \xi_{\mu+1}\zeta, \tag{46}$$

followed by a succession of type

$$\xi_\mu = \xi_{\mu+1}\eta, \tag{47}$$

secures the surface with condition 5, 2) in the form

$$\left.\begin{aligned} &\Phi_\rho(\xi_\rho, \eta_\rho, \zeta) = X_\rho(\xi_\rho, \eta_\rho, \zeta)\, E(\xi_\rho, \eta_\rho, \zeta) \\ &= [\xi_\rho^m + \eta_\rho^{p_1}\zeta^{q_1} E(\eta_\rho, \zeta)\xi_\rho^{m-2} + \ldots + \eta_\rho^{p_m}\zeta^{q_m} E(\eta_\rho, \zeta)]\, E(\xi_\rho, \eta_\rho, \zeta) \\ &\text{where for some particular term in } X_\rho, \text{ the } r\text{th}, \\ &\qquad p_r < r, \qquad q_r < r. \end{aligned}\right\} \tag{48}$$

Consider the surface (42) with the condition 5, 2), the $s$th term being

$$\bar{\eta}_\nu^{p_s}\zeta^{q_s} E(\bar{\eta}_\nu, \zeta)\xi_\nu^{m-s},$$

and suppose we apply to the surface $n$ transformations of type (46), dividing out each time the factor $\zeta^m$. The resulting term is

$$\bar{\eta}_\nu^{p_s}\zeta^{q_s - ns} E(\bar{\eta}_\nu, \zeta)\xi_{\nu+n}^{m-s}.$$

If $n$ is taken large enough, the exponent

$$q_s - ns$$

is made less than $s$, so that we have

$$s > q_s - ns \geqq 0,$$

or

$$\frac{q_s}{s} - 1 < n \leqq \frac{q_s}{s}.$$

So the term for which $\frac{q_s}{s}$ is least is among those first reached in which the new exponent of $\zeta$ is lower than $s$.

In the same way we show that, by applying a succession of transformations of type (47), the term for which $\frac{p_s}{s}$ is least is among the first lot reached for which the new exponent of $\bar{\eta}_r$ is less than $s$. But, by condition 2) in 5, $\frac{p_s}{s}$ and $\frac{q_s}{s}$ were least in the same term. So we secure the surface in form (48).

7. A further succession of quadratic transformations of type (43) as applied to the surface in form (48) will reduce the singularity. This follows at once by the reasoning in the first part of 5.

## D. — Succession of Quadratic Transformations in which there are Reversals of Type.

8. A succession of transformations in which there is a sufficient number of reversals of type will secure a surface of type (42).

If there is but a finite number of reversals, after the last one we are in the same position as at the start in 4, and the succession of transformations which follows, not having any reversal of type, will enable us to secure the condition derived by the method of 4. So we need here to consider only the possibility of an indefinitely large number of reversals of type.

In equation (37) consider any one of the coefficients

$$p_r(\eta, \zeta) = \zeta^{\lambda} \bar{p}_r(\eta, \zeta) = \zeta^{\lambda}[(\eta, \zeta)_{n_r} + (\eta, \zeta)_{n_r+1} + \cdots \cdot]$$

where

$$\bar{p}_r(\eta, 0) \not\equiv 0.$$

A transformation of type (40) will give for $\bar{p}_r$ a function from which we

take out the factor $\zeta^{n_r}$, the other factor being of degree less than $n_r$ unless the part $(\eta, \zeta)_{n_r}$ has $n_r$ equal linear factors. For, if

$$(\eta, \zeta)_{n_r} = \prod_{\rho=1}^{n_r} (\alpha_\rho \eta - \beta_\rho \zeta)$$

and not all the linear factors are equal (or linearly dependent), then the substitution

$$\eta = \zeta(\eta_1 + \delta_1)$$

gives

$$\zeta^{n_r} \prod_{\rho=1}^{n_r} (\alpha_\rho \eta_1 + \alpha_\rho \delta_1 - \beta_\rho)$$

and leaves an absolute term in any factor for which

$$\alpha_\rho \delta_1 \neq \beta_\rho,$$

thus securing in the product of the factors terms of degree less than $n_r$. Also the degree might be lowered on account of terms in some later part as $(\eta, \zeta)_{n_r+k}$. But, if all the factors of $(\eta, \zeta)_{n_r}$ are equal (or linearly dependent) and $\delta_1$ is taken so as to satisfy the condition

$$\alpha_\rho \delta_1 = \beta_\rho, \qquad \rho = 1, 2, \ldots\ldots n_r,$$

then after the factor $\zeta^{n_r}$ is divided out, we have left but one term in $\eta_1^{n_r}$, which cannot cancel with any term from another part of the function, as all later terms have as a factor some power of $\zeta$. Accordingly a succession of transformations of type (40), if it does not reduce the degree of the part not divisible by $\zeta$, must leave a term in $\eta_\nu^{n_r}$. Now when the reversal of type is first made, the $\epsilon$ of (41) is zero, as is seen by considering the use of transformation (8) § 1, 5. Then we take out a factor $\eta_\nu^{n_r}$ and leave a constant term. So a succession of transformations which contains reversals of type must reduce the degree of the function $p_r$ (possibly to zero), except for factors taken out which are powers of the $\eta$ and $\zeta$ variables. Accordingly, by a succession of transformations containing a sufficiently large number of reversals of type, the coefficient $p_r$ must be reduced to the type

$$\eta^{p_r} \zeta^{q_r} E(\eta, \zeta).$$

9. All further transformations to be considered may be taken of the types

$$\xi_\mu = \xi_{\mu+1} \zeta, \qquad \eta_\mu = \eta_{\mu+1} \zeta, \tag{49}$$

$$\xi_\mu = \xi_{\mu+1} \eta, \qquad \zeta_\mu = \zeta_{\mu+1} \eta. \tag{50}$$

For if a transformation of type (40) or (41) in which the $\delta$ or $\epsilon$ is not zero were used, we should have in all the coefficients of $X_r$ in (42), outside of the $E$ factor, only powers of one variable. Suppose it to be $\zeta$; then, by means of a succession of transformations of type (46), we can reduce some term to a form in which the exponent of $\zeta$ is less than $r$, and thus secure a reduction of singularity.

10. A sufficiently long succession of transformations of types (49) and (50), applied to surface of type (42), unless it first secures reduction of singularity, will secure the condition that, for some term (the $r$th),

$$\left.\begin{aligned} \frac{p_r}{r} &\leqq \frac{p_s}{s} \\ \frac{q_r}{r} &\leqq \frac{q_s}{s} \end{aligned}\right\}, \qquad s = 2, 3, \ldots\ldots m.$$

Consider the two terms

$$\eta^{p_s}\zeta^{q_s}E(\eta, \zeta)\xi^{m-s}, \qquad \eta^{p_t}\zeta^{q_t}E(\eta, \zeta)\xi^{m-t}$$

Any transformation of type (49) leaves the $p_s$ and $p_t$ unchanged, and increases the

$$\begin{aligned} q_s &\text{ by } p_s - s, \\ q_t &\text{ “ } p_t - t. \end{aligned}$$

Any transformation of type (50) leaves the $q_s$ and $q_t$ unchanged, but increases the

$$\begin{aligned} p_s &\text{ by } q_s - s, \\ p_t &\text{ “ } q_t - t. \end{aligned}$$

Represent

$$\left.\begin{aligned} \frac{q_r - r}{r} &\text{ by } \mathrm{K}_r, \\ \frac{p_r - r}{r} &\text{ “ } \Pi_r \end{aligned}\right\} \qquad r = 2, 3, \ldots\ldots m, \tag{51}$$

So, for each transformation of type (49) the $\mathrm{K}_r$ is increased by the $\Pi_r$, and for each transformation of type (50) the $\Pi_r$ is increased by the $\mathrm{K}_r$. We shall show that finally we must have one of two conditions

$$\begin{aligned} &a)\ \Pi_s \geqq \Pi_t, && \mathrm{K}_s \geqq \mathrm{K}_t, \\ &b)\ \Pi_s \leqq \Pi_t, && \mathrm{K}_s \leqq \mathrm{K}_t. \end{aligned}$$

Suppose, at any stage, neither of these conditions holds, and we have, for example,

$$\Pi_s > \Pi_t, \qquad \mathrm{K}_s < \mathrm{K}_t. \tag{52}$$

Then, for a transformation of type (49), supposing the new K's to be $K_s'$, $K_t'$, we have

$$K_s' = K_s + \Pi_s,$$

$$K_t' = K_t + \Pi_t,$$

and so

$$K_t' - K_s' = K_t - K_s - (\Pi_s - \Pi_t) < K_t - K_s.$$

Also, for a transformation of type (50), if the new $\Pi$'s are $\Pi_s'$ $\Pi_t'$, we have

$$\Pi_s' = \Pi_s + K_s,$$

$$\Pi_t' = \Pi_t + K_t,$$

and

$$\Pi_s' - \Pi_t' = \Pi_s - \Pi_t - (K_t - K_s) < \Pi_s - \Pi_t.$$

So when a condition of type (52) holds, any transformation applied will reduce the difference of either the $\Pi$'s or K's, if in fact it does not change the sign of the difference. Further, the reduction is each time by a value not infinitesimal, for it is at least $1/st$, as is seen by considering the values of $K_r$ and $\Pi_r$ in (51). So the succession of transformations of whatever kind must finally reduce the difference of either the $\Pi$'s or the K's to zero, or change its sign, and then we secure either condition *a*) or *b*).

When one of these conditions has once been secured, any further transformation will not change it; for, in condition *a*), a transformation of type (49) will add at least as much to the $K_s$ as to the $K_t$, and so retain the inequality of the same order, and similar conditions are seen to hold in the other cases. Also, as one of the conditions *a*) or *b*) must hold finally, whatever the pair of values $s$ and $t$, we shall have some value as $r$ such that

$$\Pi_r \leqq \Pi_s, \qquad K_r \leqq K_s, \qquad s = 2, 3, \ldots\ldots m.$$

from which follows the required condition

$$\left.\begin{aligned} \frac{p_r}{r} &\leqq \frac{p_s}{s} \\ \frac{q_r}{r} &\leqq \frac{q_s}{s} \end{aligned}\right\} \qquad s = 2, 3, \ldots\ldots m.$$

11. The method of 6, applied to the surface resulting from the treatment of 10, will secure the result of 6. It may be that already either $p_r < r$ or $q_r < r$, but in such a case the number of transformations of

type (46) or (47) can be considered zero, while in the other case we have exactly the initial conditions of 6, the result of which then can be secured in any case whatever.

12. In the case of surface (48) any succession of transformations of types (49) and (50) will finally reduce the degree of the singular point.

Consider the term

$$\eta_\rho^{p_r} \zeta^{q_r} E(\eta_\rho, \zeta) \xi_\rho^{m-r}.$$

Any transformation of type (49) adds to the exponent of $\zeta$, $p_r - r$, and as $p_r < r$, the exponent of $\zeta$ is reduced. In the same way we see that any transformation of type (50) reduces the exponent of the $\eta$ variable. So in any case, by virtue of the reduction of degree, we must have finally either

$$p_r < r - q_r \qquad \text{or} \qquad q_r < r - p_r,$$

in either of which cases the sum of the exponents of the three variables

$$(m - r) + p_r + q_r$$

is less than $m$, and we have reduction of the singularity.

## § 5.

### Parametric Representation of the Neighborhood of the Original Singular Point.

We have shown that in all cases $T$, the neighborhood of a singular point, can be mapped upon a finite number of regions $t_1, t_2, \ldots . t_\nu$ as defined in § 2, C. Apply a properly chosen transformation to each point of class 1) and repeat the operation on each set of resulting points of the same class as they are formed. We have proved also that after a finite number of operations all the resulting points of class 1) are of order lower than $m$. Then, by continuing the process, it follows that, after a finite number of transformations, all points of class 1) must disappear, and so we shall have left only regions of class 2). Each of these regions admits of representation by means of a finite number of sets of parametric formulae of type $(A)$.

Classify all the singular points which present themselves in groups as follows: —

In the first group, place the original point; in the second, all singular points derived from it by the first quadratic transformation, together with whatever auxiliary transformations accompany it; these points correspond to the singular points of the curves that represent the irreducible

factors of $\phi(\xi, \eta)$, to the points of intersection of two such curves, and to the points of class 1) in § 2, 1. In the third group place all singular points derived in a similar way from those of the second group, etc.

Suppose $n$ to be the number of the last group in which there are singular points. From what we have proved, $n$ must be finite.

The neighborhood of a point in the $n$th group is represented by the neighborhoods of a finite number of regular points, together with a finite number of regular regions, and so by a finite number of parametric formulae of type $(A)$. The neighborhood of a point in the $(n-1)$st group is represented by the neighborhoods of a finite number of points of the $n$th group, together with a finite number of regular regions, however small the neighborhoods of the singular points are taken; but as the neighborhood of any point in the $n$th group is represented by a finite number of parametric formulae of type $(A)$, the same follows for any point of the $(n-1)$st group, using the intermediate transformation to get the parametric formulae.

This reasoning can be carried on until the original singular point is reached, since the mapping of the neighborhood of the original point upon a finite number of regions of classes 1) and 2) applies to each of the later singular points also, and then furnishes the step by which we know that we can always pass from the $(\nu+1)$st to the $\nu$th group.

Thus we have the coördinates $\xi, \eta, \zeta$ of the surface

$$\Phi(\xi, \eta, \zeta) = 0$$

expressed in parametric formulae of the desired type, the parameters being in general coördinates of points of some simple surface. Then by using the intermediate transformations connecting $x, y, z$ with $\xi, \eta, \zeta$, we represent the first set of coördinates in the desired form.